

EQUIDISTRIBUTION OF VALUES OF LINEAR FORMS ON QUADRATIC SURFACES.

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ABSTRACT. In this paper we investigate the distribution of the set of values of a linear map at integer points on a quadratic surface. In particular, it is shown that subject to certain algebraic conditions, this set is equidistributed. This can be thought of as a quantitative version of the main result from [Sar11]. The methods used are based on those developed by A. Eskin, S. Mozes and G. Margulis in [EMM98]. Specifically, they rely on equidistribution properties of unipotent flows.

1. INTRODUCTION.

Consider the following situation. Let X be a rational surface in \mathbb{R}^d , R be a fixed region in \mathbb{R}^s and $F : X \rightarrow \mathbb{R}^s$ be a polynomial map. An interesting problem is to investigate the size of the set

$$Z = \{x \in X \cap \mathbb{Z}^d : F(x) \in R\},$$

consisting of integer points in X such that the corresponding values of F , are in R . Suppose that the set of values of F at the integer points of X , is dense in \mathbb{R}^s . In this case, the set Z will be infinite. However, the set

$$Z_T = \{x \in X \cap \mathbb{Z}^d : F(x) \in R, \|x\| \leq T\},$$

can be considered. This set will be finite, and its size will depend on T . Typically, the density assumption indicates that the set Z might be equidistributed, within the set of all integer points in X . Namely, as T increases, the size of the set Z_T , should be proportional to the appropriately defined volume, of the set

$$\{x \in X : F(x) \in R, \|x\| \leq T\},$$

consisting of real points on X , with values in R and bounded norm. Such a result, if it is obtained, can be seen as quantifying the denseness of the values of F at integral points.

The situation described above is too general, but it serves as motivation for what is to come. So far, what is proven, is limited to special cases. For instance, when $M : \mathbb{R}^d \rightarrow \mathbb{R}^s$ is a linear map, classical methods can be used to establish necessary and sufficient conditions, which ensure the values of M on \mathbb{Z}^d are dense in \mathbb{R}^s . The equidistribution problem described above can also be considered in this case. It is straightforward to obtain an asymptotic estimate for the number of integer points with bounded norm whose values lie in some compact region of \mathbb{R}^s (cf. [Cas72]).

When $Q : \mathbb{R}^d \rightarrow \mathbb{R}$ is a quadratic form the situation is that of the Oppenheim conjecture. In [Mar89], G. Margulis obtained necessary and sufficient conditions to ensure that the values of Q on \mathbb{Z}^d are dense in \mathbb{R} . Considerable work has gone into the equidistribution problem in this case, first by S.G. Dani and G. Margulis, who obtained an asymptotic lower bound for the number of integers with bounded height such that their images lie in a fixed interval (cf. [DM93]). Later, A. Eskin, G. Margulis and S. Mozes, gave the corresponding asymptotic upper bound for the same problem (cf. [EMM98]). The major ingredient, used in the proof of Oppenheim conjecture, is to relate the density of the values of a quadratic form at integers to the density of certain orbits inside a homogeneous space. This connection was first noted by M. S. Raghunathan in the late 70's (appearing in print in [Dan81], for instance). It is, in this way, using tools from dynamical systems to study the orbit closures of subgroups corresponding to quadratic forms, that Margulis proved the Oppenheim conjecture. Similarly, the later refinement, due to Dani-Margulis, who considered the values of quadratic forms at primitive integral

points in [DM90] and work on the equidistribution (quantitative) problem by Dani-Margulis and Eskin-Margulis-Mozes, were also obtained by studying the orbit closures of subgroups acting on homogeneous spaces.

Similar techniques were also used by A. Gorodnik in [Gor04], to study the set of values of a pair, consisting of a quadratic and linear form, at integer points and in [Sar11] to establish conditions, sufficient to ensure that the values of a linear map at integers lying on a quadratic surface are dense in the range of the map. The main result of this paper deals with the corresponding equidistribution problem and is stated in the following Theorem.

Theorem 1.1. *Suppose Q is a quadratic form on \mathbb{R}^d such that Q is non-degenerate, indefinite with rational coefficients. Let $M = (L_1, \dots, L_s) : \mathbb{R}^d \rightarrow \mathbb{R}^s$ be a linear map such that:*

- (1) *The following relations hold, $d > 2s$ and $\text{rank}(Q|_{\ker(M)}) = d - s$.*
- (2) *The quadratic form $Q|_{\ker(M)}$ has signature (r_1, r_2) where $r_1 \geq 3$ and $r_2 \geq 1$.*
- (3) *For all $\alpha \in \mathbb{R}^s \setminus \{0\}$, $\alpha_1 L_1 + \dots + \alpha_s L_s$ is non rational.*

Let $a \in \mathbb{Q}$ be such that the set $\{v \in \mathbb{Z}^d : Q(v) = a\}$ is non empty. Then there exists $C_0 > 0$ such that for every $\theta > 0$ and all compact $R \subset \mathbb{R}^s$ with piecewise smooth boundary, there exists a $T_0 > 0$ such that for all $T > T_0$,

$$(1 - \theta) C_0 \text{Vol}(R) T^{d-s-2} \leq |\{v \in \mathbb{Z}^d : Q(v) = a, M(v) \in R, \|v\| \leq T\}| \leq (1 + \theta) C_0 \text{Vol}(R) T^{d-s-2},$$

where $\text{Vol}(R)$ is the s dimensional Lebesgue measure of R .

Remark 1.2. The constant C_0 appearing in Theorem 1.1 is such that

$$C_0 \text{Vol}(R) T^{d-s-2} \sim \text{Vol}(\{v \in \mathbb{R}^d : Q(v) = a, M(v) \in R, \|v\| \leq T\})$$

where the volume on the right is the Haar measure on the surface defined by $Q(v) = a$.

Remark 1.3. Theorem 1.1 should hold with the condition that $\text{rank}(Q|_{\ker(M)}) = d - s$ replaced by the condition that $\text{rank}(Q|_{\ker(M)}) > 3$. Dealing with the more general situation requires taking into account the nontrivial unipotent part of $\text{Stab}_{SO(Q)}(M)$, as such lower bounds could probably be proved using methods of [DM93], but so far no way has been found to obtain the statement that would be needed in order to obtain an upper bound.

Remark 1.4. As in [EMM98] it would be possible to obtain a version of Theorem 1.1 where the condition that $\|v\| < T$ was replaced by $v \in TK_0$ where K_0 is an arbitrary deformation of the unit ball by a continuous and positive function. It should also be possible to obtain a version of Theorem 1.1 where the parameters T_0 and C_0 remain valid for any pair (Q, M) coming from compact subsets of pairs satisfying the conditions of the Theorem.

Remark 1.5. The cases when the quadratic form $Q|_{\ker(M)}$ has signature $(2, 2)$ or $(2, 1)$ can be considered exceptional. There are asymptotically more integers than expected (by a factor of $\log T$) lying on certain surfaces defined by quadratic forms of signature $(2, 2)$ or $(2, 1)$. This leads to counterexamples of Theorem 1.1 in the cases when the quadratic form $Q|_{\ker(M)}$ has signature $(2, 2)$ or $(2, 1)$. Details of these examples are found in Section 6.

Outline of the paper. Recall that in the proof of the quantitative Oppenheim conjecture (cf. [EMM98]) one needs to consider an unbounded function on the space of lattices. Similarly, in order to prove Theorem 1.1 one needs to consider an unbounded function F on a certain homogeneous space. The strategy is to try and apply an ergodic theorem to F in order to show that the average of the values of F evaluated along a certain orbit converges to the average of F on the entire space. This is the fact that corresponds to the fact that integral points on the quadratic surface with values in R are equidistributed. The main problem in doing this is that F is unbounded and so in order to prove such an ergodic theorem one needs precise information about the behaviour of the orbits near the cusp. This information is obtained in Section 3. The required ergodic theorem is then proved in Section 4. Finally in Section 5 the proof of Theorem 1.1 is completed using an approximation argument similar to that found in [EMM98]. Specifically, the averages of F over the space are related to the quantity

$C_0 \text{Vol}(R) T^{d-s-2}$ and the averages of F along an orbit are related to the number of integer points with bounded height, lying on the surface and with values in R . In Section 2 the basic notation is set up and the main results from Section 3 and Section 4 are stated.

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2. SET UP.

2.1. Main results. For the rest of the paper the following convention is in place: s, d and p will be fixed natural numbers such that $2s < d$ and $0 < p < d$. Also, r_1 and r_2 will be varying, natural numbers such that $d - s = r_1 + r_2$. Let \mathcal{L} denote the space of linear forms on \mathbb{R}^d and let \mathcal{C}_{Lin} denote the subset of \mathcal{L}^s such that for all $M \in \mathcal{C}_{\text{Lin}}$ condition 3 of Theorem 1.1 is satisfied. A quadratic form on \mathbb{R}^d is said to be defined over \mathbb{Q} , if it has rational coefficients or is a scalar multiple of a form with rational coefficients. For a , a rational number let $\mathcal{Q}(p, a)$ denote quadratic forms on \mathbb{R}^d defined over \mathbb{Q} with signature $(p, d - p)$ such that the set $\{v \in \mathbb{Z}^d : Q(v) = a\}$ is non empty for all $Q \in \mathcal{Q}(p, a)$. Define

$$\mathcal{C}_{\text{Pairs}}(a, r_1, r_2) = \{(Q, M) : Q \in \mathcal{Q}(p, a), M \in \mathcal{C}_{\text{Lin}} \text{ and } Q|_{\ker(M)} \text{ has signature } (r_1, r_2)\}.$$

Note that for $r_1 \geq 3$ and $r_2 \geq 1$ the set $\mathcal{C}_{\text{Pairs}}(a, r_1, r_2)$ consists of pairs satisfying the conditions of Theorem 1.1. Although the set $\mathcal{C}_{\text{Pairs}}(a, r_1, r_2)$ and hence its subsets and sets derived from them depend on a , this dependence is not a crucial one, so from now on, most of the time this dependence will be omitted from the notation. For $M \in \mathcal{L}^s$ and $R \subset \mathbb{R}^s$ a connected region with smooth boundary let $V_M(R) = \{v \in \mathbb{R}^d : M(v) \in R\}$. For $Q \in \mathcal{Q}(p, d - p)$, $a \in \mathbb{Q}$ and $\mathbb{K} = \mathbb{R}$ or \mathbb{Z} let $X_Q^a(\mathbb{K}) = \{v \in \mathbb{K}^d : Q(v) = a\}$. Denote the annular region inside \mathbb{R}^d by $A(T_1, T_2) = \{v \in \mathbb{R}^d : T_1 \leq \|v\| \leq T_2\}$. Using this notation, we state the following (equivalent) version of Theorem 1.1, which will be proved in Section 5.

Theorem 2.1. *Suppose that $r_1 \geq 3$, $r_2 \geq 1$ and $a \in \mathbb{Q}$. Then for all $(Q, M) \in \mathcal{C}_{\text{Pairs}}(a, r_1, r_2)$ there exists $C_0 > 0$ such that for every $\theta > 0$ and all compact $R \subset \mathbb{R}^s$ with piecewise smooth boundary, there exists a $T_0 > 0$ such that for all $T > T_0$,*

$$(1 - \theta) C_0 \text{Vol}(R) T^{d-s-2} \leq |X_Q^a(\mathbb{Z}) \cap V_M(R) \cap A(0, T)| \leq (1 + \theta) C_0 \text{Vol}(R) T^{d-s-2}.$$

Remark 2.2. As remarked previously, the cases when $r_1 = 2$ and $r_2 = 2$ or $r_1 = 2$ and $r_2 = 1$ are interesting. In dimensions 3 and 4 there can be more integer points than expected lying on some surfaces defined by quadratic forms of signature $(2, 2)$ or $(2, 1)$, this means that the statement of Theorem 2.1 fails for certain pairs. In Section 6 these counterexamples are explicitly constructed. Moreover, it is shown that this set of pairs is big in the sense that it is of second category. We note that as in [EMM98] one could also show that this set has measure zero and one could prove the expected asymptotic formula as in Theorem 2.1 for almost all pairs.

Even though Theorem 2.1 fails when $r_1 = 2$ and $r_2 = 2$ or $r_1 = 2$ and $r_2 = 1$, we do have the following uniform upper bound, which will be proved in Section 5 and is analogous to Theorem 2.3 from [EMM98].

Theorem 2.3. *Suppose $r_1 \geq 3$, $r_2 \geq 1$ and $a \in \mathbb{Q}$. Let $R \subset \mathbb{R}^s$ be a compact region with piecewise smooth boundary. Then, for all $(Q, M) \in \mathcal{C}_{\text{Pairs}}(a, r_1, r_2)$, there exists a constant C depending only on (Q, M) and R such that for all $T > 1$,*

$$|X_Q^a(\mathbb{Z}) \cap V_M(R) \cap A(0, T)| \leq CT^{d-s-2}.$$

For all $(Q, M) \in \mathcal{C}_{\text{Pairs}}(a, 2, 2)$ there exists a constant C depending only on (Q, M) and R such that for all $T > 2$,

$$(2.1) \quad |X_Q^a(\mathbb{Z}) \cap V_M(R) \cap A(0, T)| \leq C(\log T) T^{d-s-2}.$$

If $s = 1$, then for all $(Q, M) \in \mathcal{C}_{\text{Pairs}}(a, 2, 1)$ there exists a constant C depending only on (Q, M) and R such that for all $T > 2$ (2.1) holds. If $s = 2$, then for all $(Q, M) \in \mathcal{C}_{\text{Pairs}}(a, 2, 1)$ such that

$\ker(M) \cap \mathbb{Z}^5 = 0$ there exists a constant C depending only on (Q, M) and R such that for all $T > 2$ (2.1) holds.

Remark 2.4. The extra condition, $\ker(M) \cap \mathbb{Z}^5 = 0$, appearing in the case when $(Q, M) \in \mathcal{C}_{\text{Pairs}}(a, 2, 1)$ and $s = 2$ is probably not necessary but is needed to deal with limitations arising in the proof of Theorem 2.5 for this case.

2.2. A canonical form. For $v_1, v_2 \in \mathbb{R}^d$ we will use the notation $\langle v_1, v_2 \rangle$ to denote the standard inner product in \mathbb{R}^d . For a set of vectors $v_1, \dots, v_i \in \mathbb{R}^d$ we will also use the notation $\langle v_1, \dots, v_i \rangle$ to denote the span of v_1, \dots, v_i in \mathbb{R}^d , although this could lead to some ambiguity, the meaning of the notation should be clear from the context.

For some computations it will be convenient to know that our system is conjugate to a canonical form. Let e_1, \dots, e_d be the standard basis of \mathbb{R}^d . Let (Q_0, M_0) be the pair consisting of a quadratic form and a linear map defined by

$$Q_0(v) = Q_{1, \dots, s}(v) + 2v_{s+1}v_d + \sum_{i=s+2}^{s+r_1} v_i^2 - \sum_{i=s+r_1+1}^{d-1} v_i^2 \quad \text{and} \quad M_0(v) = (v_1, \dots, v_s),$$

where $v_i = \langle v, e_i \rangle$ and $Q_{1, \dots, s}(v)$ is a non degenerate quadratic form in variables v_1, \dots, v_s . By Lemma 2.2 of [Sar11] all pairs (Q, M) such that $\text{rank}(Q|_{\ker(M)}) = d - s$ and the signature of $Q|_{\ker(M)}$ is (r_1, r_2) are equivalent to the pair (Q_0, M_0) in the sense that there exists $g_d \in GL_d(\mathbb{R})$ and $g_s \in GL_s(\mathbb{R})$ such that $(Q, M) = (Q_0^{g_d}, g_s M_0^{g_d})$, where for $g \in GL_d(\mathbb{R})$ we write $Q = Q_0^g$ if and only if $Q_0(gv) = Q(v)$ for all $v \in \mathbb{R}^d$. Moreover since $R \subset \mathbb{R}^s$ is arbitrary, up to rescaling and possibly replacing R by $g_s R$ we assume that $g_d \in SL_d(\mathbb{R})$ and that g_s is the identity. Let

$$\mathcal{C}_{\text{SL}}(a, r_1, r_2) = \{g \in SL_d(\mathbb{R}) : (Q_0^g, M_0^g) \in \mathcal{C}_{\text{Pairs}}(a, r_1, r_2)\}.$$

For $g \in \mathcal{C}_{\text{SL}}(a, r_1, r_2)$, let G_g be the identity component of the group $\{x \in SL_d(\mathbb{R}) : Q_0^g(xv) = Q_0^g(v)\}$, $\Gamma_g = G_g \cap SL_d(\mathbb{Z})$, $H_g = \{x \in G_g : M_0^g(xv) = M_0^g(v)\}$ and $K_g = H_g \cap g^{-1}O_d(\mathbb{R})g$. By examining the description of the subgroup H_g , given in Section 2.3 of [Sar11] it is clear that K_g is a maximal compact subgroup of H_g . It is a standard fact that G_g is a connected semisimple Lie group and hence, has no nontrivial rational characters. Therefore, because Q_0^g is defined over \mathbb{Q} , the Borel Harish-Chandra Theorem (cf. [PR94], Theorem 4.13) implies Γ_g is a lattice in G_g . We will consider the dynamical system that arises from H_g acting on G_g/Γ_g . For $\mathbb{K} = \mathbb{R}$ or \mathbb{Z} , the shorthand $X_{Q_0^g}^a(\mathbb{K}) = X_g(\mathbb{K})$ will be used.

2.3. Equidistribution of measures. We will consider the function α , as defined in [EMM98]. It is an unbounded function on the space of unimodular lattices in \mathbb{R}^d . It has the properties that it can be used to bound certain functions that we will consider and it is left K_I invariant. Similar functions have been considered in [Sch95] where it is related to various quantities involving successive minima of a lattice. Let Δ be a lattice in \mathbb{R}^d . For any such Δ we say that a subspace U of \mathbb{R}^d is Δ -rational if $\text{Vol}(U/U \cap \Delta) < \infty$. Let

$$\Psi_i(\Delta) = \{U : U \text{ is a } \Delta\text{-rational subspace of } \mathbb{R}^d \text{ with } \dim U = i\}.$$

For $U \in \Psi_i(\Delta)$ define $d_\Delta(U) = \text{Vol}(U/U \cap \Delta)$. Note that $d_\Delta(U) = \|u_1 \wedge \dots \wedge u_i\|$ where u_1, \dots, u_i is a basis for $U \cap \Delta$ over \mathbb{Z} and the norm on $\bigwedge^i(\mathbb{R}^d)$ is induced from the euclidean norm on \mathbb{R}^d . If N_i denotes a norm on $\bigwedge^i(\mathbb{R}^d)$ we can generalise the definition of the function d_Δ by defining it with respect to the norm N_i . This is done as follows.

For $U \in \Psi_i(\Delta)$ define $d_\Delta^{N_i}(U) = N_i(u_1 \wedge \dots \wedge u_i)$ where u_1, \dots, u_i form a basis for $U \cap \Delta$ over \mathbb{Z} . Now we can define the function α^N , as follows

$$\alpha_i^N(\Delta) = \sup_{U \in \Psi_i(\Delta)} \frac{1}{d_\Delta^{N_i}(U)} \quad \text{and} \quad \alpha^N(\Delta) = \max_{1 \leq i \leq d} \alpha_i^N(\Delta).$$

If the norm N_i denotes the standard norm on $\bigwedge^i(\mathbb{R}^d)$ then it will be omitted from the notation. Because N_i is a norm these alpha functions are equivalent in the sense that there exists constants

n_1 and n_2 such that

$$n_1 \alpha_i^N(\Delta) \leq \alpha_i(\Delta) \leq n_2 \alpha_i^N(\Delta),$$

for all lattices Δ . In (2.3) and Theorem 2.7 we consider α as a function on G_g/Γ_g , this is done via the canonical embedding of G_g/Γ_g into the space of unimodular lattices in \mathbb{R}^d , given by $x\Gamma_g \rightarrow x\mathbb{Z}^d$. Specifically, every $x \in G_g/\Gamma_g$ can be identified with its image under this embedding before applying α to it. For $f \in C_c(\mathbb{R}^d)$ and $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ we define the function $F_{f,g} : G_g/\Gamma_g \rightarrow \mathbb{R}$ by

$$(2.2) \quad F_{f,g}(x) = \sum_{v \in X_g(\mathbb{Z})} f(xv).$$

The function α has the property that there exists a constant $c(f)$ depending only on the support and maximum of f such that for all x in G_g/Γ_g ,

$$(2.3) \quad F_{f,g}(x) \leq c(f) \alpha(x).$$

The last property is well known and follows from Minkowski's Theorem on successive minima, see Lemma 2 of [Sch68] for example. Alternatively, see [HW08] for an up to date review of many related results.

We will be carrying out integration on various measure spaces defined by the groups introduced at the beginning of the section. With this in mind let us introduce the following notation for the corresponding measures. If v denotes some variable, the notation dv is used to denote integration with respect to Lebesgue measure and this variable. Let μ_g be the Haar measure on G_g/Γ_g , if $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ then since Γ_g is a lattice in G_g we can normalise so that $\mu_g(G_g/\Gamma_g) = 1$. In addition, ν_g will denote the measure on K_g normalised so that $\nu_g(K_g) = 1$. Let m_g^a denote the Haar measure on $X_g^a(\mathbb{R})$ defined by

$$(2.4) \quad \int_{\mathbb{R}^d} f(v) dv = \int_{-\infty}^{\infty} \int_{X_g^a(\mathbb{R})} f(v) dm_g^a(v) da.$$

The following Theorem provides us with our upper bounds and will be proved in Section 3.

Theorem 2.5. *Let $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ be arbitrary and let $\Delta = g\mathbb{Z}^d$. Let $\{a_t : t \in \mathbb{R}\}$ denote a self adjoint one parameter subgroup of $SO(2, 1)$ embedded into H_I so that it fixes the subspace $\langle e_{s+2}, \dots, e_{d-1} \rangle$ and only has eigenvalues $e^{-t}, 1$ and e^t .*

(I) *Suppose $r_1 \geq 3$, $r_2 \geq 1$ and $0 < \delta < 2$ or $r_1 \geq 2$, $r_1 \geq 1$ and $0 < \delta < 1$, then*

$$\sup_{t > 0} \int_{K_I} \alpha(a_t k \Delta)^\delta d\nu_I(k) < \infty.$$

(II) *Suppose $r_1 = r_2 = 2$ or $r_1 = 2$, $r_2 = 1$ and $d = 4$, then*

$$\sup_{t > 1} \frac{1}{t} \int_{K_I} \alpha(a_t k \Delta) d\nu_I(k) < \infty.$$

(III) *Suppose $r_1 = 2$, $r_2 = 1$, $d = 5$ and $\langle e_3, e_4, e_5 \rangle \cap \Delta = 0$, then*

$$\sup_{t > 1} \frac{1}{t} \int_{K_I} \alpha(a_t k \Delta) d\nu_I(k) < \infty.$$

Remark 2.6. The extra condition, $\langle e_3, e_4, e_5 \rangle \cap \Delta = 0$, appearing in the part III, corresponds to extra condition appearing in Theorem 2.3.

In Section 4 we will modify the results from Section 4 of [EMM98] and combine them with Theorem 2.5 to prove the following Theorem which will be a major ingredient of the proof of Theorem 2.1.

Theorem 2.7. *Suppose $r_1 \geq 3$ and $r_2 \geq 1$. Let $A = \{a_t : t \in \mathbb{R}\}$ be a one parameter subgroup of H_g , not contained in any proper normal subgroup of H_g , such that there exists a continuous homomorphism $\rho : SL_2(\mathbb{R}) \rightarrow H_g$ with $\rho(D) = A$ and $\rho(SO(2)) \subset K_g$ where $D = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0 \right\}$. Let $\phi \in L^1(G_g/\Gamma_g)$ be a continuous function such that for some $0 < \delta < 2$ and some $C > 0$,*

$$(2.5) \quad |\phi(\Delta)| < C \alpha(\Delta)^\delta, \text{ for all } \Delta \in G_g/\Gamma_g.$$

Then for all $\epsilon > 0$ and all $g \in \mathcal{C}_{SL}(r_1, r_2)$ there exists $T_0 > 0$ such that for all $t > T_0$,

$$\left| \int_{K_g} \phi(a_t k) d\nu_g(k) - \int_{G_g/\Gamma_g} \phi d\mu_g \right| \leq \epsilon.$$

Remark 2.8. The condition that A should not be contained in any proper normal subgroup of H_g is only necessary in the case when $H_g \cong SO(2, 2)$, since in all other cases H_g is simple.

3. THE ALPHA FUNCTION AND UPPER BOUNDS.

In this section we prove Theorem 2.5 by making necessary modifications to Section 5 from [EMM98]. The strategy is to prove that a system of inequalities, which can be used to prove Theorem 2.5, are valid for the functions α_i . The system of inequalities is completely analogous to that given in Lemma 5.7 of [EMM98] and is stated below as Propositions 3.1 and 3.2. The main difference is that in the present case, we must deal with the fact that H_I does not act irreducibly on \mathbb{R}^d . By definition $H_I \cong SO(r_1, r_2)$ and fixes $\langle e_1, \dots, e_s \rangle$. Let $\{a_t : t \in \mathbb{R}\}$ denote a self adjoint one parameter subgroup of $SO(2, 1)$ embedded into H_I so that it fixes the subspace $\langle e_{s+2}, \dots, e_{d-1} \rangle$. Moreover suppose that the only eigenvalues of a_t are e^{-t} , 1 and e^t .

Proposition 3.1. *Let $\Delta = g\mathbb{Z}^d$ for some $g \in \mathcal{C}_{SL}(r_1, r_2)$. If either of the following are satisfied,*

- (1) $0 < \delta < 1$, $c > 0$, $r_1 \geq 2$, $r_2 \geq 1$ and $0 < i < d$ are arbitrary,
- (2) $0 < \delta < 2$ and $c > 0$ are arbitrary and
 - (a) $r_1 \geq 3$, $r_2 \geq 1$ and $0 < i < d$ are arbitrary or,
 - (b) $r_1 = r_2 = 2$ and $i = 1$ or $d - 1$ or,
 - (c) $r_1 = 2$, $r_2 = 1$, $i = 1$ or $d - 1$ and $\Delta \cap \langle e_{s+1}, \dots, e_d \rangle = 0$.

Then exists $t > 0$ and $w > 1$ such that for all $h \in H_I$,

$$(3.1) \quad \int_{K_I} \alpha_i(a_t k h \Delta)^\delta d\nu_I(k) < c \alpha_i(h \Delta)^\delta + w^2 \max_{0 < j \leq \min\{d-i, i\}} (\alpha_{i+j}(h \Delta) \alpha_{i-j}(h \Delta))^{\delta/2}.$$

In the case when $(r_1, r_2) = (2, 2)$, note that $SO(2, 2)$ is locally isomorphic to $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. The isomorphism is given as the action of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ on $M_2(\mathbb{R}) \cong \mathbb{R}^4$ defined by $(g_1, g_2)m = g_1^t m g_2$ which preserves the determinant of m which is a quadratic form of signature $(2, 2)$ on \mathbb{R}^4 . There is a basis so that the embedding of $SO(2, 1)$ becomes locally isomorphic to the diagonal subgroup of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ and a_t becomes $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \times \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. In Proposition 3.2 the subgroup \hat{K}_I is just the standard maximal compact subgroup of H_I in the case when $r_1 = 2$ and $r_2 = 1$. In the case when $r_1 = r_2 = 2$ we define \hat{K}_I to be a maximal compact subgroup of the diagonally embedded copy of $SO(2, 1)$ under the above identifications. Moreover, define $\hat{\nu}_I$ to be the Haar measure on \hat{K}_I normalised so that $\hat{\nu}_I(\hat{K}_I) = 1$.

Proposition 3.2. *Let $\Delta = g\mathbb{Z}^d$ for some $g \in \mathcal{C}_{SL}(r_1, r_2)$. If either of the following are satisfied,*

- (1) $r_1 = 2$, $r_2 = 1$ and $0 < i < d$ are arbitrary,
- (2) $r_1 = r_2 = 2$, and $1 < i < d - 1$ is arbitrary.

Then there exists $w(t) > 1$ and a norm, N_i on $\bigwedge^i(\mathbb{R}^d)$, such that for all $t > 0$ and all $h \in H_I$,

$$(3.2) \quad \int_{\hat{K}_I} \alpha_i^N(a_t k h \Delta) d\hat{\nu}_I(k) < \alpha_i^N(h \Delta) + w(t)^2 \max_{0 < j \leq \min\{d-i, i\}} (\alpha_{i+j}(h \Delta) \alpha_{i-j}(h \Delta))^{1/2}.$$

The following notations will be in place for the rest of the section. Let $g \in \mathcal{C}_{SL}(r_1, r_2)$ be arbitrary, let $\Delta = g\mathbb{Z}^d$, let $L_{i,\Delta}$ be the subspace of dimension i such that $\alpha_i(\Delta) = 1/d_\Delta(L_{i,\Delta})$. Let $v_1, \dots, v_i \in \mathbb{R}^d$ be such that $\Delta \cap L_{i,\Delta} = \langle v_1, \dots, v_i \rangle_{\mathbb{Z}}$ and let $v_{i,\Delta} = v_1 \wedge \dots \wedge v_i \in \bigwedge^i(\mathbb{R}^d)$. Let $\rho_{i,d} : H_I \rightarrow GL\left(\bigwedge^i(\mathbb{R}^d)\right)$ be the i^{th} exterior representation. There is a decomposition of the form

$$\bigwedge^i(\mathbb{R}^d) = \bigoplus_{j=1}^{\mathcal{N}(i,d,s)} U_j,$$

where each subspace listed in the direct sum is $\rho_{i,d}(H_I)$ invariant. Moreover,

$$(U_j, \rho_{i,d}(H_I)|_{U_j}) \cong \left(\bigwedge^k (\mathbb{R}^{r_1+r_2}), \rho_{k,(r_1+r_2)}(SO(r_1, r_2)) \right),$$

for some $1 \leq k \leq \min\{i, r_1 + r_2\}$. We consider the action $H_I \times \bigwedge^i(\mathbb{R}^d) \rightarrow \bigwedge^i(\mathbb{R}^d)$ given by $(h, v) \rightarrow \rho_{i,d}(h)v$, or equivalently $(h, v_1 \wedge \dots \wedge v_i) \rightarrow hv_1 \wedge \dots \wedge hv_i$. From now on we will denote this action by hv . Let $\mathcal{F}_{i,d}(H_I) = \{v \in \bigwedge^i(\mathbb{R}^d) : H_I v = v\}$ be the the fixed vectors of $\rho_{i,d}(H_I)$ and $\pi_{i,d} : \bigwedge^i(\mathbb{R}^d) \rightarrow \mathcal{F}_{i,d}(H_I)$ be the orthogonal projection. The following observation means that for the rest of the section we can suppose that $v_{i,\Delta} \notin \mathcal{F}_{i,d}(H_I)$.

Lemma 3.3. *If $v_{i,\Delta} \in \mathcal{F}_{i,d}(H_I)$ then the conclusions of Proposition 3.1 and Proposition 3.2 hold.*

Proof. It is easy to see that if $v_{i,\Delta} \in \mathcal{F}_{i,d}(H_I)$ then for any $t > 0$ and $h \in H_I$,

$$\int_{K_I} \alpha_i(a_t k h \Delta)^\delta d\nu_I(k) = \alpha_i(\Delta)^\delta.$$

Moreover, this implies that the conclusion of Proposition 3.1 holds because we can take

$$w = \min_{0 < j \leq \min\{d-i, i\}} (\alpha_i(\Delta) / \alpha_{i+j}(\Delta) \alpha_{i-j}(\Delta))^{\delta/2}.$$

It is also clear that the conclusions of Proposition 3.2 also hold in this case. \square

The following simple Lemma is needed only for the case when $r_1 = 2$, $r_1 = 1$ and $d = 5$.

Lemma 3.4. *The condition that $\Delta \cap \langle e_{s+1}, \dots, e_d \rangle = 0$ implies that $\pi_{1,d}(v_{1,\Delta}) \neq 0$.*

Proof. Suppose $\Delta \cap \langle e_{s+1}, \dots, e_d \rangle = 0$. By definition $v_{1,\Delta} \in \Delta$. However, if $\pi_{1,d}(v_{1,\Delta}) = 0$ then $v_{1,\Delta} \in \langle e_{s+1}, \dots, e_d \rangle$ which is a contradiction. \square

3.1. Proof of Propositions 3.1 and 3.2. The main idea is that if N_i denotes a norm on $\bigwedge^i(\mathbb{R}^d)$ then Propositions 3.1 and 3.2 follow from statements of the form

$$(3.3) \quad \int_{K_I} N_i(a_t k h v_{i,\Delta})^{-\delta} d\nu_I(k) < c N_i(h v_{i,\Delta})^{-\delta}$$

where the c, δ, h and t come with appropriate quantifiers. We use the decomposition of $\bigwedge^i(\mathbb{R}^d)$ described above to define a norm, specifically the norm that we use is defined to be the maximum of the norms on each invariant subspace listed in the decomposition. Except for the case when the subspace is in $\mathcal{F}_{i,d}(H_I)$ we can then use the results of [EMM98] to obtain our conclusions. The first Lemma to be proved deals with the case when $\pi_{i,d}(v_{i,\Delta}) \neq 0$, in this case, for all $h \in H_I$ there is an absolute lower bound for $\|a_t k h v_{i,\Delta}\|^\delta$ and this gives a strong result.

Lemma 3.5. *Suppose that $\pi_{i,d}(v_{i,\Delta}) \neq 0$. Then for all $\delta > 0$ and all $h \in H_I$,*

$$\lim_{t \rightarrow \infty} \int_{K_I} \|a_t k h v_{i,\Delta}\|^{-\delta} d\nu_I(k) = 0.$$

Proof. Consider the action of a_t on $\bigwedge^i(\mathbb{R}^d)$ as previously defined. Let W^+ , W^0 and W^- be (respectively) expanding, neutral and contracting eigenspaces for this action. Let $h \in H_I$ be arbitrary. Let $p^+ : \bigwedge^i(\mathbb{R}^d) \rightarrow W^+$ be the orthogonal projection and let $\mathcal{K}_\eta = \{k \in K_I : \|p^+(k h v_{i,\Delta})\| \leq \eta\}$. Our choice of how $\{a_t : t \in \mathbb{R}\}$ is embedded into H_I means that $\rho_{i,d}(K_I) v_{i,\Delta} \not\subseteq W^0 \oplus W^-$, this and the fact that $v_{i,\Delta} \notin \mathcal{F}_{i,d}(H_I)$, means that $\mathcal{K}_\eta \neq K_I$. Moreover, \mathcal{K}_η is an algebraic subvariety of K_I and consequently $\nu_I(\mathcal{K}_\eta) = 0$. Because $\pi_{i,d}(v_{i,\Delta}) \neq 0$, the previous observations mean that for all $\delta > 0$,

$$(3.4) \quad \lim_{\eta \rightarrow 0} \|\pi_{i,d}(v_{i,\Delta})\|^{-\delta} \nu_I(\mathcal{K}_\eta) = 0.$$

Note that $\|a_t k h v_{i,\Delta}\| \geq e^t \|p^+(k h v_{i,\Delta})\|$ and that $\|a_t k h v_{i,\Delta}\| \geq \|\pi_{i,d}(v_{i,\Delta})\|$ and hence for $\eta > 0$,

$$\begin{aligned} \int_{K_I} \|a_t k h v_{i,\Delta}\|^{-\delta} d\nu_I(k) &= \int_{K_I \setminus \mathcal{K}_\eta} \|a_t k h v_{i,\Delta}\|^{-\delta} d\nu_I(k) + \int_{\mathcal{K}_\eta} \|a_t k h v_{i,\Delta}\|^{-\delta} d\nu_I(k) \\ &\leq e^{-\delta t} \eta^{-\delta} + \|\pi_{i,d}(v_{i,\Delta})\|^{-\delta} \nu_I(\mathcal{K}_\eta). \end{aligned}$$

Therefore, setting $\eta = e^{-t/2\delta}$ and using (3.4) we get the conclusion of the Lemma. \square

The next two Lemmas provide the bounds necessary to prove Propositions 3.1 and 3.2. In Lemma 3.6 the cases when $\pi_{i,d}(v_{i,\Delta}) = 0$ and $\pi_{i,d}(v_{i,\Delta}) \neq 0$ are considered separately. The first case is dealt with by our choice of norm and results from [EMM98]. In the second case, Lemma 3.5 can be used to complete the proof of Lemma 3.6. Lemma 3.7 is proved by considering special norms defined by combining the definitions of norms given in [EMM98] and the norm we consider in Lemma 3.6.

Lemma 3.6. *If the parameters δ, r_1, r_2, i and c satisfy either of the conditions 1 or 2 from Proposition 3.1, then there exists $t_0 > 0$ such that for all $t > t_0$ and all $h \in H_I$,*

$$(3.5) \quad \int_{K_I} \|a_t k h v_{i,\Delta}\|^{-\delta} d\nu_I(k) < c \|h v_{i,\Delta}\|^{-\delta}.$$

Proof. For $1 \leq j \leq \mathcal{N}(i, d, s)$ let $\tau_j : \bigwedge^i(\mathbb{R}^d) \rightarrow U_j$ be the orthogonal projections. Consider the following norm on $\bigwedge^i(\mathbb{R}^d)$ given by

$$N_i(v) = \max_{1 \leq j \leq \mathcal{N}(i, d, s)} \|\tau_j(v)\|.$$

The key fact that is used in the following is that

$$(3.6) \quad \int_{K_I} \|a_t k h v_{i,\Delta}\|^{-\delta} d\nu_I(k) \leq \int_{B_j} \|a_t k \tau_j(h v_{i,\Delta})\|^{-\delta} d\nu^j(k),$$

where B_j denotes the maximal compact subgroup of $\rho_{i,d}(H_I)|_{U_j}$ and ν^j denotes the corresponding Haar measure. Suppose $\pi_{i,d}(h v_{i,\Delta}) = \pi_{i,d}(v_{i,\Delta}) = 0$. It follows from (3.6) and Proposition 5.4 of [EMM98], that if the parameters δ, r_1, r_2, i and c satisfy any of the conditions 1, 2a or 2b from Proposition 3.1, then for all $1 \leq j \leq \mathcal{N}(i, d, s)$, there exists $t_0 > 0$ such that for all $t > t_0$ and $h \in H_I$,

$$\int_{K_I} \|a_t k h v_{i,\Delta}\|^{-\delta} d\nu_I(k) \leq c \|\tau_j(h v_{i,\Delta})\|^{-\delta}.$$

Hence

$$\begin{aligned} \int_{K_I} N_i(a_t k h v_{i,\Delta})^{-\delta} d\nu_I(k) &\leq \min_{1 \leq j \leq \mathcal{N}(i, d, s)} \left(\int_{K_I} \|\tau_j(a_t k h v_{i,\Delta})\|^{-\delta} d\nu_I(k) \right) \\ &\leq c \min_{1 \leq j \leq \mathcal{N}(i, d, s)} \left(\|\tau_j(h v_{i,\Delta})\|^{-\delta} \right) = c N_i(h v_{i,\Delta})^{-\delta}. \end{aligned}$$

Therefore if the parameters δ, r_1, r_2, i and c satisfy any of the conditions 1, 2a or 2b from Proposition 3.1 and $\pi_{i,d}(v_{i,\Delta}) = 0$ the conclusion of the Lemma follows. However, if $\pi_{i,d}(v_{i,\Delta}) \neq 0$ then the conclusion of the Lemma follows from Lemma 3.5. If conditions 2c of Proposition 3.1 are satisfied we use Lemma 3.4 to show that $\pi_{1,d}(v_{1,\Delta}) \neq 0$ then the conclusion of the Lemma follows from Lemma 3.5. \square

Lemma 3.7. *If the parameters δ, r_1, r_2, i and c satisfy either of the conditions 1 or 2 from Proposition 3.2, then there exists a norm, N on $\bigwedge^i(\mathbb{R}^d)$, such that for all $t > 0$ and $h \in H_I$,*

$$\int_{\widehat{K}_I} N(a_t k h v_{i,\Delta})^{-1} d\hat{\nu}_I(k) \leq N(h v_{i,\Delta})^{-1}.$$

Proof. Let $h \in H_I$ be arbitrary. Suppose that $r_1 = 2$ and $r_2 = 1$. For $v \in \mathbb{R}^3$ let $\|v\|_* = \max\left\{\sqrt{v_1^2 + v_2^2}, |v_3|\right\}$. Let N_i, τ_j and U_j be as before, each U_j carries a norm $\|v\|_j = \|v\|_*$ if $U_j = \mathbb{R}^3$,

or $\|v\|_j = \|v\|$ if $U_j = \mathbb{R}$. Define $N_i(v) = \max_{1 \leq j \leq \mathcal{N}(i,d,s)} \|\tau_j(v)\|_j$. Lemma 5.5 of [EMM98] implies that if $U_j = \mathbb{R}^3$ then for all $t > 0$ and $h \in H_I$,

$$(3.7) \quad \int_{K_I} \|\tau_j(a_t k h v_{i,\Delta})\|_j^{-1} d\nu_I(k) \leq \|\tau_j(h v_{i,\Delta})\|_j^{-1}.$$

It is clear that if $U_j = \mathbb{R}$ then for all $t > 0$, (3.7) holds. Then (3.7) implies that for all $t > 0$ and $h \in H_I$,

$$(3.8) \quad \begin{aligned} \int_{K_I} N_i(a_t k h v_{i,\Delta})^{-1} d\nu_I(k) &\leq \min_{1 \leq j \leq \mathcal{N}(i,d,s)} \left(\int_{K_I} \|\tau_j(a_t k h v_{i,\Delta})\|_j^{-1} d\nu_I(k) \right) \\ &\leq \min_{1 \leq j \leq \mathcal{N}(i,d,s)} \left(\|\tau_j(h v_{i,\Delta})\|_j^{-1} \right) = N_i(h v_{i,\Delta})^{-1}. \end{aligned}$$

This proves the Lemma for the case when δ, r_1, r_2, i and c satisfy conditions 1 from Proposition 3.2.

Suppose that $r_1 = r_2 = 2$. Recall that $\Lambda^2(\mathbb{R}^4) = V_1 \oplus V_2$ where each V_i is a three dimensional invariant subspace for the action of $\rho_{2,4}(SO(2,2))$. On each subspace $SO(2,2)$ preserves a quadratic form of signature $(2,1)$. Hence, on each V_i we can define the norm $\|\cdot\|_*$. For $i = 1$ or 2 , let $\sigma_i : \Lambda^2(\mathbb{R}^4) \rightarrow V_i$ denote the orthogonal projection. For $v \in \Lambda^2(\mathbb{R}^4)$, define $\|v\|_2^\# = \max_{i=1,2} \|\sigma_i(v)\|_*$. Let U_j be as before, each U_j carries a norm, $\|v\|_j = \|v\|^\#$ if $U_j = \Lambda^2(\mathbb{R}^4)$ or $\|v\|_j = \|v\|$ if $U_j = \mathbb{R}$. If $U_j = \mathbb{R}^4$ then it is clear that $\mathbb{R}^4 = W_1 \oplus W_2$ where W_1 and W_2 are \hat{K}_I invariant subspaces with $\dim(W_1) = 1$ and $\dim(W_2) = 3$. For $i = 1$ or 2 , let $\iota_i : \mathbb{R}^4 \rightarrow W_i$ denote the orthogonal projections. In this case define $\|v\|_j = \max\{\|\iota_1(v)\|, \|\iota_2(v)\|_*\}$. Define $N_i(v) = \max_{1 \leq j \leq \mathcal{N}(i,d,s)} \|\tau_j(v)\|_j$. Suppose $U_j = \mathbb{R}^4$, then from the definition of the norm

$$(3.9) \quad \begin{aligned} \int_{\hat{K}_I} \|\tau_j(a_t \hat{k} h v_{i,\Delta})\|_j^{-1} d\hat{\nu}_I(\hat{k}) &\leq \\ \min \left\{ \int_{\hat{K}_I} \|\iota_1 \tau_j(a_t \hat{k} h v_{i,\Delta})\|^{-1} d\hat{\nu}_I(\hat{k}), \int_{\hat{K}_I} \|\iota_2 \tau_j(a_t \hat{k} h v_{i,\Delta})\|_*^{-1} d\hat{\nu}_I(\hat{k}) \right\}. \end{aligned}$$

It is clear that for all $t \in \mathbb{R}$ and $h \in H_I$,

$$(3.10) \quad \int_{\hat{K}_I} \|\iota_1 \tau_j(a_t \hat{k} h v_{i,\Delta})\|^{-1} d\hat{\nu}_I(\hat{k}) = \|\iota_1 \tau_j(h v_{i,\Delta})\|^{-1}.$$

Lemma 5.5 from [EMM98] implies that for all $t > 0$ and $h \in H_I$,

$$(3.11) \quad \int_{\hat{K}_I} \|\iota_2 \tau_j(a_t \hat{k} h v_{i,\Delta})\|_*^{-1} d\hat{\nu}_I(\hat{k}) \leq \|\iota_2 \tau_j(a_t h v_{i,\Delta})\|_*^{-1}.$$

It follows from (3.9), (3.10) and (3.11) that for all $t > 0$ and $h \in H_I$,

$$(3.12) \quad \int_{\hat{K}_I} \|\tau_j(a_t \hat{k} h v_{i,\Delta})\|_j^{-1} d\hat{\nu}_I(\hat{k}) \leq \|\tau_j(h v_{i,\Delta})\|_j^{-1}.$$

It is clear that if $U_j = \mathbb{R}$, then for all $t \in \mathbb{R}$, (3.12) holds. Lemma 5.9 from [EMM98] implies that if $U_j = \Lambda^2(\mathbb{R}^4)$, then for all $t > 0$, (3.12) also holds. Since (3.12) holds for all $1 \leq j \leq \mathcal{N}(i,d,s)$, by the same method as used to obtain (3.8), it implies that for all $t > 0$,

$$\int_{\hat{K}_I} N_i(a_t \hat{k} h v_{i,\Delta})^{-1} d\hat{\nu}_I(\hat{k}) \leq N_i(h v_{i,\Delta})^{-1}.$$

This proves the Lemma for the case when δ, r_1, r_2, i and c satisfy conditions 2 from Proposition 3.2. \square

As noted at the beginning of the section Propositions 3.1 and 3.2 follow from Lemmas 3.6 and 3.7. This is done in the same way as Lemma 5.7 is obtained in [EMM98]. For example we prove Proposition 3.1 below. The proof of Proposition 3.2 is identical, except one should use Lemma 3.7 in place of Lemma 3.6.

Proof of Proposition 3.1. Let $h \in H_I$ be arbitrary. Suppose conditions 1 or 2 from Proposition 3.1 hold. Since $a_t k h v_1, \dots, a_t k h v_i$ form a basis for $a_t k h L_{i,\Delta} \cap a_t k h \Delta$ we have $d_{a_t k h \Delta}(a_t k h L_{i,\Delta}) = \|a_t k h(v_1 \wedge \dots \wedge v_i)\| = \|a_t k h v_{i,\Delta}\|$. Hence Lemma 3.6 implies that for all $c > 0$ there exists a $t_0 > 0$ such that for all $t > t_0$ and $h \in H_I$,

$$(3.13) \quad \int_{K_I} d_{a_t k h \Delta}(a_t k h L_{i,\Delta})^{-\delta} d\nu_I(k) < c d_{h\Delta}(h L_{i,\Delta})^{-\delta} \leq c \alpha_i(h\Delta)^\delta.$$

Let $w(t) = \max_{0 < j < d} \|\rho_{j,d}(a_t)\|$. Then for all $v \in \bigwedge^i(\mathbb{R}^d)$ and $0 < j < d$,

$$(3.14) \quad w(t)^{-1} \leq \|a_t v\| / \|v\| \leq w(t).$$

Let $\Upsilon_i(h, \Delta, w(t)) = \{L \in \Psi_i(\Delta) : d_{h\Delta}(hL) < w(t)^2 d_{h\Delta}(h L_{i,\Delta})\}$. If $L \in \Psi_i(\Delta) \setminus \Upsilon_i(h, \Delta, w(t))$ then $w(t)^2 d_{h\Delta}(h L_{i,\Delta}) \leq d_{h\Delta}(hL)$. Equation (3.14) implies that for all $t > 0$ we have $w(t)^{-1} d_{h\Delta}(hL) \leq d_{a_t k h \Delta}(a_t k h L)$ and $d_{a_t k h \Delta}(a_t k h L_{i,\Delta}) \leq w(t) d_{h\Delta}(h L_{i,\Delta})$ and hence for all $k \in K_I$,

$$(3.15) \quad d_{a_t k h \Delta}(a_t k h L_{i,\Delta}) < d_{a_t k h \Delta}(a_t k h L).$$

If $\Upsilon_i(h, \Delta, w(t)) = \{h L_{i,\Delta}\}$, then for $L \in \Psi_i(\Delta)$ such that $L \neq h L_{i,\Delta}$ then $L \in \Psi_i(\Delta) \setminus \Upsilon_i(h, \Delta, w(t))$. It follows from (3.13) and (3.15) and the definition of α_i that in this case for all $c > 0$ there exists $t_0 > 0$ such that for all $t > t_0$ and $h \in H_I$,

$$(3.16) \quad \begin{aligned} \int_{K_I} \alpha_i(a_t k h \Delta)^\delta d\nu_I(k) &= \int_{K_I} \sup_{L \in \Psi_i(\Delta)} \left(d_{a_t k h \Delta}(a_t k h L)^{-\delta} \right) d\nu_I(k) \\ &\leq \int_{K_I} d_{a_t k h \Delta}(a_t k h L_{i,\Delta})^{-\delta} d\nu_I(k) < c \alpha_i(h\Delta)^\delta. \end{aligned}$$

Assume now that $\Upsilon_i(h, \Delta, w(t)) \neq \{h L_{i,\Delta}\}$. Let $V \in \Upsilon_i(h, \Delta, w(t))$ with $V \neq h L_{i,\Delta}$. Then $\dim(V + h L_{i,\Delta}) = i + j$ and $\dim(V \cap h L_{i,\Delta}) = i - j$ for $j > 0$. Now by (3.14),

$$(3.17) \quad \alpha_i(a_t k h \Delta) < w(t) \alpha_i(h\Delta).$$

Note that

$$(3.18) \quad \alpha_i(h\Delta) = d_{h\Delta}(L_{i,h\Delta})^{-1}.$$

Because $V \in \Upsilon_i(h, \Delta, w(t))$, $d_{h\Delta}(V) < w(t)^2 d_{h\Delta}(L_{i,h\Delta})$ and hence

$$(3.19) \quad w(t) d_{h\Delta}(L_{i,h\Delta})^{-1} = \sqrt{w(t)^2 d_{h\Delta}(L_{i,h\Delta})^{-2}} < w(t)^2 (d_{h\Delta}(L_{i,h\Delta}) d_{h\Delta}(V))^{-1/2}.$$

Lemma 5.6 from [EMM98] implies that

$$(3.20) \quad d_{h\Delta}(L_{i,h\Delta} \cap V) d_{h\Delta}(L_{i,h\Delta} + V) \leq d_{h\Delta}(L_{i,h\Delta}) d_{h\Delta}(V)$$

and hence, combining (3.17), (3.18), (3.19) and (3.20) we get that, for all $k \in K_I$ and $t > 0$ and $h \in H_I$,

$$(3.21) \quad \alpha_i(a_t k h \Delta) \leq w(t)^2 \sqrt{\alpha_{i+j}(h\Delta) \alpha_{i-j}(h\Delta)}.$$

By combining (3.21) and (3.16) it is clear that (3.1) holds. \square

3.2. Proof of Theorem 2.5 parts II and III. Theorem 2.5 part I is proved by following the method of [EMM98]. Since there are more modifications needed to prove parts II and III we present proofs of these cases below. Note that part I of Theorem 2.5 is required in the proof of part II for the case when $r_1 = 2$ and $r_1 = 1$. Therefore it is assumed that we have already proved part I.

Proof of Theorem 2.5 parts II and III. Let $f_i(h) = \alpha_i^N(h\Delta)$ where N_i are the norms so that (3.2) is satisfied. We will be applying Lemma 5.13 from [EMM98] to the functions f_i (or a weighted average of them), to justify this we make the following observations:

- (1) Because $\|hv_{i,\Delta}\| / \|v_{i,\Delta}\| \leq \left\| \bigwedge^i(h) \right\|$ for all $h \in H_I$, it is true that for all $\epsilon > 0$ there exists a neighbourhood $V(\epsilon)$ of 1 in H_I such that $(1 - \epsilon)f_i(h) < f_i(uh) < (1 + \epsilon)f_i(h)$ for all $h \in H_I$ and $u \in V(\epsilon)$.
- (2) Since α_i^N is left K_I invariant so is f_i .
- (3) There exists $\beta > 0$ such that $f_i(1) = \alpha_i^N(\Delta) < \beta$.

We also note that if we define $\hat{f}_i(h) = \int_{K_I} f_i(hk) d\nu_I(k)$ then \hat{f}_i also satisfies conditions 1, 2 and 3. Also, note that $\hat{f}_i(h) = \int_{K_I} \hat{f}_i(hk) d\nu_I(k) = \hat{f}_i(h)$ because ν_I is a probability measure. Suppose $r_1 = 2$ and $r_2 = 1$ and $d = 4$. By Proposition 3.2, for $0 < i < 4$, there exists a $w(t) > 1$ such that for all $h \in H_I$ and $t > 0$,

$$(3.22) \quad \int_{K_I} f_i(a_t k h) d\nu_I(k) < f_i(h) + w(t)^2 \max_{0 < j \leq \min\{d-i, i\}} (f_{i+j}(h) f_{i-j}(h))^{1/2}.$$

Writing out the inequalities explicitly gives

$$(3.23) \quad \int_{K_I} f_2(a_t k h) d\nu_I(k) < f_2(h) + w(t)^2 (f_1(h) f_3(h))^{1/2},$$

and

$$(3.24) \quad \int_{K_I} f_i(a_t k h) d\nu_I(k) < f_i(h) + w(t)^2 (f_2(h))^{1/2},$$

for $i = 1$ or 3 . Let $\hat{f}_i(h) = \int_{K_I} f_i(hk) d\nu_I(k)$. Writing out the inequalities (3.23) and (3.24) with h replaced by hk and integrating over $k \in K_I$ gives

$$(3.25) \quad \int_{K_I} \hat{f}_2(a_t k h) d\nu_I(k) < \hat{f}_2(h) + w(t)^2 (\hat{f}_1(h) + \hat{f}_3(h)),$$

and

$$(3.26) \quad \int_{K_I} \hat{f}_i(a_t k h) d\nu_I(k) < \hat{f}_i(h) + w(t)^2 \int_{K_I} (f_2(hk))^{1/2} d\nu_I(k),$$

for $i = 1$ or 3 . Since we may write $h \in H_I$ as $k' a_t k''$ for $k', k'' \in K_I$ we have that

$$(3.27) \quad \int_{K_I} (f_2(hk))^{1/2} d\nu_I(k) < \infty,$$

by part I of Theorem 2.5. Therefore combining (3.27) and (3.26) we see that, for $i = 1$ or 3 , \hat{f}_i satisfies the conditions of Lemma 5.13 of [EMM98] and so

$$(3.28) \quad \int_{K_I} \hat{f}_i(a_t k) d\nu_I(k) < Bt,$$

for $i = 1$ or 3 . Writing (3.25) with h replaced by hk and integrating over $k \in K_I$ again gives

$$(3.29) \quad \int_{K_I} \hat{f}_2(a_t k h) d\nu_I(k) < \hat{f}_2(h) + w(t)^2 \int_{K_I} (\hat{f}_1(hk) + \hat{f}_3(hk)) d\nu_I(k).$$

Hence, using (3.28) and Lemma 5.13 of [EMM98] we see that

$$(3.30) \quad \int_{K_I} \hat{f}_2(a_t k) d\nu_I(k) < Bt.$$

From (3.28) we see that for $i = 1$ or 3 ,

$$\int_{K_I} \alpha_i^N(a_t k \Delta) d\nu_I(k) = \int_{K_I} \int_{K_I} \alpha_i^N(a_t k' k \Delta) d\nu_I(k) d\nu_I(k') = \int_{K_I} \hat{f}_i(a_t k') d\nu_I(k') < Bt,$$

and for $i = 2$,

$$\int_{K_I} \alpha_2^N(a_t k \Delta) d\nu_I(k) = \int_{K_I} \int_{K_I} \int_{K_I} \alpha_2^N(a_t k'' k' k \Delta) d\nu_I(k) d\nu_I(k') d\nu_I(k'') = \int_{K_I} \hat{f}_i(a_t k'') d\nu_I(k'') < Bt.$$

This proves part II of Theorem 2.5 for the case when $r_1 = 2$, $r_1 = 1$ and $d = 4$.

Suppose $r_1 = r_2 = 2$ or $r_1 = 2, r_2 = 1, d = 5$ and $\Delta \cap \langle e_3, e_4, e_5 \rangle = 0$. Let $f_i(h) = \alpha_i^N(h\Delta)$ where N_i are the norms so that (3.2) is satisfied. By Proposition 3.1 parts 2b and 2c for $i = 1$ or $d - 1$ and all $c > 0$, there exists a $t_0 > 0$ and $w_0 > 1$ such that for all $h \in H_I$,

$$(3.31) \quad \int_{K_I} f_i(a_{t_0}kh) d\nu_I(k) < cf_i(h) + w_0^2 \max_{0 < j \leq \min\{d-i, i\}} (f_{i+j}(h) f_{i-j}(h))^{1/2}.$$

By Proposition 3.2 for $1 < i < d - 1$, there exists a $w(t) > 1$ such that for all $h \in H_I$ and $t > 0$,

$$(3.32) \quad \int_{\widehat{K}_I} f_i(a_tkh) d\hat{\nu}_I(k) < f_i(h) + w(t)^2 \max_{0 < j \leq \min\{d-i, i\}} (f_{i+j}(h) f_{i-j}(h))^{1/2}.$$

Let $q(i) = i(d - i)$. Then from (3.31) and (3.32) we get for all $0 < \epsilon < 1$, arbitrary $c_i > 0$ for $i = 1$ or $d - 1$ and $c_i = 1$ for $1 < i < d - 1$, there exists $t_0 > 0$ and $w = \max\{w(t_0), w_0\} > 1$ such that

$$(3.33) \quad \int_{\widehat{K}_I} \epsilon^{q(i)} f_i(a_{t_0}kh) d\hat{\nu}_I(k) < c_i \epsilon^{q(i)} f_i(h) + \epsilon w^2 \max_{0 < j \leq \min\{d-i, i\}} \left(\epsilon^{q(i+j)} f_{i+j}(h) \epsilon^{q(i-j)} f_{i-j}(h) \right)^{1/2}.$$

Integrating (3.33) over K_I with k replaced by hk implies that

$$(3.34) \quad \int_{\widehat{K}_I} \epsilon^{q(i)} \hat{f}_i(a_{t_0}kh) d\hat{\nu}_I(k) < c_i \epsilon^{q(i)} \hat{f}_i(h) + \epsilon w^2 \max_{0 < j \leq \min\{d-i, i\}} \left(\epsilon^{q(i+j)} \hat{f}_{i+j}(h) + \epsilon^{q(i-j)} \hat{f}_{i-j}(h) \right).$$

Let $f_\epsilon = \sum_{0 \leq i \leq d} \epsilon^{q(i)} \hat{f}_i$. Since $\epsilon^{q(i)} \hat{f}_i < f_\epsilon$ and $\hat{f}_0 = \hat{f}_d = 1$, by summing (3.34) we obtain

$$(3.35) \quad \int_{\widehat{K}_I} f_\epsilon(a_{t_0}kh) d\hat{\nu}_I(k) < c' f_\epsilon(h) + 2d\epsilon w^2 f_\epsilon(h),$$

for some $0 < c' < 1$. Take $\epsilon = (1 - c')/2dw^2$ and we see that

$$(3.36) \quad \int_{\widehat{K}_I} f_\epsilon(a_{t_0}kh) d\hat{\nu}_I(k) < f_\epsilon(h).$$

Now it remains to note that f_ϵ restricted to a the copy \widehat{K}_I inside $SO(2, 2)$ satisfies the conditions of Lemma 5.13 of [EMM98] and hence there exists $B > 0$, so that for all $t > 1$,

$$(3.37) \quad \int_{\widehat{K}_I} f_\epsilon(a_tk) d\hat{\nu}_I(k) < Bt.$$

Since

$$\begin{aligned} \int_{K_I} \alpha_i^N(a_tk\Delta) d\nu_I(k) &= \int_{\widehat{K}_I} \int_{K_I} \alpha_i^N(a_t\hat{k}k\Delta) d\nu_I(k) d\hat{\nu}_I(\hat{k}) \\ &= \int_{\widehat{K}_I} \hat{f}_i(a_t\hat{k}) d\hat{\nu}_I(\hat{k}) < \epsilon^{-q(i)} \int_{\widehat{K}_I} f_\epsilon(a_t\hat{k}) d\hat{\nu}_I(\hat{k}), \end{aligned}$$

(3.37) implies the conclusions of part II of Theorem 2.5 when $r_1 = r_2 = 2$ and part III Theorem 2.5 when $r_1 = 2$ and $r_2 = 1$. \square

4. ERGODIC THEOREMS.

For subgroups W_1 and W_2 of G_g , let $X(W_1, W_2) = \{g \in G_g : W_2g \subset gW_1\}$. As in [EMM98] the ergodic theory is based on Theorem 3 from [DM93] reproduced below in a form relevant to the current situation.

Theorem 4.1. *Suppose $r_1 \geq 2$ and $r_2 \geq 1$. Let $g \in \mathcal{C}_{SL}(r_1, r_2)$ be arbitrary. Let $U = \{u_t : t \in \mathbb{R}\}$ be a unipotent one parameter subgroup of G_g and ϕ be a bounded continuous function on G_g/Γ_g . Let \mathcal{D} be a compact subset of G_g/Γ_g and let $\epsilon > 0$ be given. Then there exists finitely many proper closed subgroups H_1, \dots, H_k of G_g , such that $H_i \cap \Gamma_g$ is a lattice in H_i for all i , and compact subsets C_1, \dots, C_k of*

$X(H_1, U), \dots, X(H_k, U)$ respectively such that for all compact subsets F of $\mathcal{D} - \bigcup_{1 \leq i \leq k} C_i \Gamma_g / \Gamma_g$ there exists a $T_0 > 0$ such that for all $x \in F$ and $T > T_0$,

$$\left| \frac{1}{T} \int_0^T \phi(u_t x) dt - \int_{G_g / \Gamma_g} \phi d\mu_g \right| < \epsilon.$$

Remark 4.2. By construction the subgroups H_i occurring are such that $H_i \cap \Gamma_g$ is Zariski dense in H_i and hence H_i are defined over \mathbb{Q} . For a precise reference see Theorem 3.6.2 and Remark 3.4.2 of [KSS02].

The next result is a reworking of Theorem 4.3 from [EMM98]. The difference is that in Lemma 4.3 the identity is fixed as the base point for the flow and the condition that H_g be maximal is dropped.

Lemma 4.3. *Suppose $r_1 \geq 2$ and $r_2 \geq 1$. Let $g \in \mathcal{C}_{SL}(r_1, r_2)$ be arbitrary. Let $U = \{u_t : t \in \mathbb{R}\}$ be a one parameter unipotent subgroup of H_g , not contained in any proper normal subgroup of H_g . Let ϕ be a bounded continuous function on G_g / Γ_g . Then for all $\epsilon > 0$ and $\eta > 0$ there exists a $T_0 > 0$ such that for all $T > T_0$,*

$$(4.1) \quad \nu_g \left(\left\{ k \in K_g : \left| \frac{1}{T} \int_0^T \phi(u_t k) dt - \int_{G_g / \Gamma_g} \phi d\mu_g \right| > \epsilon \right\} \right) \leq \eta.$$

Proof. Let H_1, \dots, H_k and C_1, \dots, C_k be as in Theorem 4.1. Let $\gamma \in \Gamma_g$, consider $Y_i(\gamma) = K_g \cap X(H_i, U)\gamma$. Suppose that $Y_i(\gamma) = K_g$, then $Uk\gamma^{-1} \subset k\gamma^{-1}H_i$ for all $k \in K_g$. In other words

$$(4.2) \quad k^{-1}Uk \subset \gamma^{-1}H_i\gamma \quad \text{for all } k \in K_g.$$

The subgroup $\langle k^{-1}Uk : k \in K_g \rangle$ is normalised by $U \cup K_g$ and clearly $\langle k^{-1}Uk : k \in K_g \rangle \subseteq \langle U \cup K_g \rangle \subseteq H_g$. If G is a simple Lie group with finite centre, with maximal compact subgroup K , it follows from exercise A.3, chapter IV of [Hel01] that K is also a maximal proper subgroup of G . This means that because H_g is semisimple with finite centre, any connected subgroup L of H_g containing K_g can be represented as $L = H'K_g$ where H' is a connected normal subgroup of H_g . Because U is not contained in any proper normal subgroup of H_g , this implies that $\langle U \cup K_g \rangle = H_g$. Therefore, $\langle k^{-1}Uk : k \in K_g \rangle$ is a normal subgroup of H_g and because U is not contained in any proper normal subgroup of H_g , we have $\langle k^{-1}Uk : k \in K_g \rangle = H_g$. This and (4.2) imply that $H_g \subset \gamma^{-1}H_i\gamma$. Note that $\gamma \in SL_d(\mathbb{Z})$ and by Remark 4.2, H_i is defined over \mathbb{Q} . Therefore, $\gamma^{-1}H_i\gamma$ is defined over \mathbb{Q} , it follows from Theorem 7.7 of [PR94] that $\gamma^{-1}H_i\gamma \cap SL_d(\mathbb{Q}) = \gamma^{-1}H_i\gamma$. Therefore Lemma 3.7 and Proposition 4.1 of [Sar11] imply that $\gamma^{-1}H_i\gamma = G_g$ which is a contradiction and therefore $Y_i(\gamma) \subsetneq K_g$. This means for all $1 \leq i \leq k$, $Y_i(\gamma)$ is a submanifold of strictly smaller dimension than K_g and hence

$$(4.3) \quad \nu_g(Y_i(\gamma)) = 0.$$

Note that because $C_i \subseteq X(H_i, U)$,

$$K_g \cap \bigcup_{1 \leq i \leq k} C_i \Gamma_g \subseteq K_g \cap \bigcup_{1 \leq i \leq k} X(H_i, U) \Gamma_g = \bigcup_{1 \leq i \leq k} \bigcup_{\gamma \in \Gamma_g} Y_i(\gamma)$$

and therefore (4.3) implies

$$(4.4) \quad \nu_g \left(K_g \cap \bigcup_{1 \leq i \leq k} C_i \Gamma_g \right) = 0.$$

Let \mathcal{D} be a compact subset of G_g such that $K_g \subset \mathcal{D}$. Then from (4.4), it follows that, for all $\eta > 0$ there exists a compact subset F of $\mathcal{D} - \bigcup_{1 \leq i \leq k} C_i \Gamma_g$, such that

$$(4.5) \quad \nu_g(F \cap K_g) \geq 1 - \eta.$$

From Theorem 4.1, for all $\epsilon > 0$ there exists a $T_0 > 0$, such that for all $x \in (F \cap K_g) / \Gamma_g$ and $T > T_0$,

$$\left| \frac{1}{T} \int_0^T \phi(u_t x) dt - \int_{G_g / \Gamma_g} \phi d\mu_g \right| < \epsilon.$$

Therefore if $k \in K_g$, $T > T_0$ and

$$\left| \frac{1}{T} \int_0^T \phi(u_t k) dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right| > \epsilon,$$

then $k \in K_g \setminus F$, but $\nu_g(K_g \setminus F) \leq \eta$ by (4.5) and this implies (4.1). \square

Lemma 4.4. *Suppose $r_1 \geq 2$ and $r_2 \geq 1$. Let $g \in \mathcal{C}_{SL}(r_1, r_2)$ be arbitrary. Let $U = \{u_t : t \in \mathbb{R}\}$ be a one parameter unipotent subgroup of H_g , not contained in any proper normal subgroup of H_g . Let ϕ be a bounded continuous function on G_g/Γ_g . Then for all $\epsilon > 0$ and $\delta > 0$ there exists a $T_0 > 0$ such that for all $T > T_0$,*

$$\left| \frac{1}{\delta T} \int_T^{(1+\delta)T} \int_{K_g} \phi(u_t k) d\nu_g(k) dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right| < \epsilon.$$

Proof. Let ϕ be a bounded continuous function on G_g/Γ_g . Lemma 4.3 implies for all $\epsilon > 0$, $\eta > 0$ and $d > 0$ there exists a $T_0 > 0$ such that for all $T > T_0$,

$$(4.6) \quad \nu_g \left(\left\{ k \in K_g : \left| \frac{1}{dT} \int_0^{dT} \phi(u_t k) dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right| > \epsilon \right\} \right) \leq \eta.$$

Using (4.6) with $d = 1$ and $d = 1 + \delta$ we get that for all $\epsilon > 0$ and $\eta > 0$ there exists a subset $\mathcal{C} \subseteq K_g$ with $\nu_g(\mathcal{C}) \geq 1 - \eta$ such that for all $k \in \mathcal{C}$ the following holds

$$\left| \int_0^T \phi(u_t k) dt - T \int_{G_g/\Gamma_g} \phi d\mu_g \right| < \epsilon T \quad \text{and} \quad \left| \int_0^{(1+\delta)T} \phi(u_t k) dt - (1+\delta)T \int_{G_g/\Gamma_g} \phi d\mu_g \right| < (1+\delta)T\epsilon.$$

Hence for all $k \in \mathcal{C}$ we have

$$\begin{aligned} & \left| \int_T^{(1+\delta)T} \phi(u_t k) dt - \delta T \int_{G_g/\Gamma_g} \phi d\mu_g \right| \\ &= \left| \int_0^{(1+\delta)T} \phi(u_t k) dt - (1+\delta)T \int_{G_g/\Gamma_g} \phi d\mu_g - \int_0^T \phi(u_t k) dt + T \int_{G_g/\Gamma_g} \phi d\mu_g \right| \\ &\leq \left| \int_0^T \phi(u_t k) dt - T \int_{G_g/\Gamma_g} \phi d\mu_g \right| + \left| \int_0^{(1+\delta)T} \phi(u_t k) dt - (1+\delta)T \int_{G_g/\Gamma_g} \phi d\mu_g \right| \\ &\leq (2+\delta)T\epsilon. \end{aligned}$$

This means that for all $\delta > 0$, $\eta > 0$ and $\epsilon > 0$,

$$\nu_g \left(\left\{ k \in K_g : \left| \frac{1}{\delta T} \int_T^{(1+\delta)T} \phi(u_t k) dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right| < \frac{(2+\delta)\epsilon}{\delta} \right\} \right) \geq 1 - \eta.$$

Since we can make ϵ and η as small as we wish this implies the claim. \square

Lemma 4.5. *Suppose $r_1 \geq 2$ and $r_2 \geq 1$. Let $A = \{a_t : t \in \mathbb{R}\}$ be a one parameter subgroup of H_g , not contained in any proper normal subgroup of H_g , such that there exists a continuous homomorphism $\rho : SL_2(\mathbb{R}) \rightarrow H_g$ with $\rho(D) = A$ and $\rho(SO(2)) \subset K_g$ where $D = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0 \right\}$. Let ϕ be a continuous function on G_g/Γ_g vanishing outside of a compact set. Then for all $g \in \mathcal{C}_{SL}(r_1, r_2)$ and $\epsilon > 0$ there exists $T_0 > 0$ such that for all $t > T_0$,*

$$\left| \int_{K_g} \phi(a_t k) d\nu_g(k) - \int_{G_g/\Gamma_g} \phi d\mu_g \right| \leq \epsilon.$$

Proof. This is very similar to the proof of Theorem 4.4 from [EMM98] and some details will be omitted. Fix $\epsilon > 0$. Assume that ϕ is uniformly continuous. Let $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then it is clear that $d_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = b_t u_t k_t w$, where $b_t = (1 + t^{-2})^{-1/2} \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1+t^{-2} \end{pmatrix}$ and $k_t = (1 + t^{-2})^{-1/2} \begin{pmatrix} 1 & t^{-1} \\ -t^{-1} & 1 \end{pmatrix}$. By our assumptions on A there exists a continuous homomorphism $\rho : SL_2(\mathbb{R}) \rightarrow H_g$ such that $\rho(D) = A$ and $\rho(SO(2)) \subset K_g$. Let $\rho(d_t) = d'_t$, $\rho(b_t) = b'_t$, $\rho(k_t) = k'_t$ and $\rho(w) = w'$. Then for all $t > 0$ and $g \in \mathcal{C}_{SL}(r_1, r_2)$,

$$(4.7) \quad \begin{aligned} \int_{K_g} \phi(d'_t k) d\nu_g(k) &= \int_{K_g} \phi(b'_t u'_t k'_t w' k) d\nu_g(k) \\ &= \int_{K_g} \phi(b'_t u'_t k) d\nu_g(k), \end{aligned}$$

since k'_t and $w' \in K_g$. It follows from (4.7) that for $r, t > 0$,

$$(4.8) \quad \begin{aligned} &\left| \int_{K_g} \phi(d'_t k) d\nu_g(k) - \int_{K_g} \phi(u'_{rt} k) d\nu_g(k) \right| \\ &\leq \left| \int_{K_g} (\phi(d'_{rt} k) - \phi(d'_t k)) d\nu_g(k) \right| + \left| \int_{K_g} (\phi(d'_{rt} k) - \phi(u'_{rt} k)) d\nu_g(k) \right| \\ &= \left| \int_{K_g} (\phi(d'_r d'_t k) - \phi(d'_t k)) d\nu_g(k) \right| + \left| \int_{K_g} (\phi(b'_{rt} u'_{rt} k) - \phi(u'_{rt} k)) d\nu_g(k) \right|. \end{aligned}$$

By uniform continuity, the fact that $\lim_{t \rightarrow \infty} b_t = I$ and (4.8) imply there exists $T_1 > 0$ and $\delta > 0$ such that for $t > T_1$ and $|r - 1| < \delta$ we have

$$\left| \int_{K_g} \phi(d'_t k) d\nu_g(k) - \int_{K_g} \phi(u'_{rt} k) d\nu_g(k) \right| \leq \epsilon.$$

Thus, if $T > T_1$ then

$$(4.9) \quad \left| \int_{K_g} \phi(d'_t k) d\nu_g(k) - \frac{1}{\delta T} \int_T^{(1+\delta)T} \int_{K_g} \phi(u'_t k) d\nu_g(k) dt \right| \leq \epsilon.$$

Combining (4.9) with Lemma 4.5 via the triangle inequality finishes the proof of the Lemma. \square

The section is completed by the proof of the main ergodic result whose proof follows that of Theorem 3.5 in [EMM98].

Proof of Theorem 2.7. Assume that ϕ is non-negative. Let $A(r) = \{x \in G_g/\Gamma_g : \alpha(x) > r\}$. Choose a continuous non-negative function g_r on G_g/Γ_g such that $g_r(x) = 1$ if $x \in A(r+1)$, $g_r(x) = 0$ if $x \notin A(r)$ and $0 \leq g_r(x) \leq 1$ if $x \in A(r) \setminus A(r+1)$. Then

$$(4.10) \quad \int_{K_g} \phi(a_t k) d\nu_g(k) = \int_{K_g} \phi(a_t k) g_r(a_t k) d\nu_g(k) + \int_{K_g} (\phi(a_t k) - \phi(a_t k) g_r(a_t k)) d\nu_g(k).$$

Let $\beta = 2 - \delta$ then for $x \in G_g/\Gamma_g$,

$$\begin{aligned} \phi(x) g_r(x) &\leq C \alpha(x)^{2-\beta} g_r(x) \\ &= C \alpha(x)^{2-\beta/2} g_r(x) \alpha(x)^{-\beta/2} \leq C r^{-\beta/2} \alpha(x)^{2-\beta/2}. \end{aligned}$$

The last inequality is true because $g_r(x) = 0$ if $\alpha(x) \leq r$. Therefore

$$(4.11) \quad \int_{K_g} \phi(a_t k) g_r(a_t k) d\nu_g(k) \leq C r^{-\beta/2} \int_{K_g} \alpha(a_t k)^{2-\beta/2} d\nu_g(k).$$

Since $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$, $r_1 \geq 3$ and $r_2 \geq 1$ Theorem 2.5 part I implies there exists B such that

$$\int_{K_g} \alpha(a_t k)^{2-\beta/2} d\nu_g(k) = \int_{K_I} \alpha \circ g^{-1}(a_t k g)^{2-\beta/2} d\nu_I(k) \leq c(g) \int_{K_I} \alpha(a_t k g)^{2-\beta/2} d\nu_I(k) < B$$

for all $t \geq 0$. Then (4.11) implies that

$$(4.12) \quad \int_{K_g} \phi(a_t k) g_r(a_t k) d\nu_g(k) \leq B C r^{-\beta/2}.$$

For all $\epsilon > 0$ there exists a compact subset, \mathcal{C} of G_g/Γ_g such that $\mu_g(\mathcal{C}) \geq 1 - \epsilon$. The function α is bounded on \mathcal{C} and hence for all $\epsilon > 0$,

$$\lim_{r \rightarrow \infty} \mu_g(A(r)) = \lim_{r \rightarrow \infty} (\mu_g(\{x \in \mathcal{C} : \alpha(x) > r\}) + \mu_g(\{x \in (G_g/\Gamma_g) \setminus \mathcal{C} : \alpha(x) > r\})) \leq \epsilon.$$

This means that

$$(4.13) \quad \lim_{r \rightarrow \infty} \mu_g(A(r)) = 0.$$

Note that

$$(4.14) \quad \int_{G_g/\Gamma_g} \phi(x) g_r(x) d\mu_g(x) \leq \int_{A(r)} \phi(x) d\mu_g(x).$$

Since $\phi \in L^1(G_g/\Gamma_g)$, (4.13) and (4.14) imply that

$$(4.15) \quad \lim_{r \rightarrow \infty} \int_{G_g/\Gamma_g} \phi(x) g_r(x) d\mu_g(x) = 0.$$

Since the function $\phi(x) - \phi(x) g_r(x)$ is continuous and has compact support, Lemma 4.5 implies for all $\epsilon > 0$ and $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ there exists $T_0 > 0$ such that for all $t > T_0$,

$$(4.16) \quad \left| \int_{K_g} (\phi(a_t k) - \phi(a_t k) g_r(a_t k)) d\nu_g(k) - \int_{G_g/\Gamma_g} (\phi(x) - \phi(x) g_r(x)) d\mu_g(x) \right| < \frac{\epsilon}{2}.$$

It is straight forward to check that (4.10), (4.12), (4.15) and (4.16) imply the conclusion of the Theorem if r is sufficiently large. \square

5. PROOF OF THEOREM 2.1.

The proof of Theorem 2.1 follows the same route as that of Sections 3.4-3.5 of [EMM98]. The main modification we make in order to handle the present situation is that we work inside the surface $X_g(\mathbb{R})$ rather than in the whole of \mathbb{R}^d . For $t \in \mathbb{R}$ and $v \in \mathbb{R}^d$ define a linear map a_t by

$$a_t v = (v_1, \dots, v_s, e^{-t} v_{s+1}, v_{s+2}, \dots, e^t v_d).$$

Note that the one parameter group $\{\hat{a}_t : t \in \mathbb{R}\} = g^{-1}\{a_t : t \in \mathbb{R}\}g \subset H_g$ and that there exists a continuous homomorphism $\rho : SL_2(\mathbb{R}) \rightarrow H_g$ with $\rho(D) = \{\hat{a}_t : t \in \mathbb{R}\}$ and $\rho(SO(2)) \subset K_g$ where $D = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0 \right\}$. Moreover note that $\{a_t : t \in \mathbb{R}\}$ is self adjoint, not contained in any normal subgroup of H_g and the only eigenvalues of a_t are e^{-t} , 1 and e^t . In other words, $\{\hat{a}_t : t \in \mathbb{R}\}$ satisfies the conditions of Theorem 2.7 and Theorem 2.5. For any natural number n , let S^{n-1} denote the unit sphere in a n dimensional Euclidean space and let $\gamma_n = \text{Vol}(S^n)$ and $c_{r_1, r_2} = \gamma_{r_1-1} \gamma_{r_2-1}$ then define

$$(5.1) \quad C_1 = c_{r_1, r_2} 2^{(2-r_1-r_2)/2} = c_{r_1, r_2} 2^{(2-d+s)/2}.$$

5.1. Proof of Theorem 2.3. In Lemma 5.1 it is shown that it is possible to approximate certain integrals over K_g by integrals over \mathbb{R}^{d-s-2} . The integral over \mathbb{R}^{d-s-2} can be used like the characteristic function of $R \times A(T/2, T)$, in particular Theorem 2.3 is proved as an application of Lemma 5.1. It should be noted that Lemma 5.1 is analogous to Lemma 3.6 from [EMM98] and its proof is similar.

Lemma 5.1. *Let f be a continuous function of compact support on $\mathbb{R}_+^d = \{v \in \mathbb{R}^d : \langle v, e_{s+1} \rangle > 0\}$ and for $g \in \mathcal{C}_{SL}(r_1, r_2)$ let*

$$J_{f,g}(\ell_1, \dots, \ell_s, r) = \frac{1}{r^{d-s-2}} \int_{\mathbb{R}^{d-s-2}} f(\ell_1, \dots, \ell_s, r, v_{s+2}, \dots, v_{d-1}, v_d) dv_{s+2} \dots dv_{d-1},$$

where $v_d = (a - Q_0^g(\ell_1, \dots, \ell_s, 0, v_{s+2}, \dots, v_{d-1}, 0)) / 2r$, so that $Q_0^g(\ell_1, \dots, \ell_s, r, v_{s+2}, \dots, v_{d-1}, v_d) = a$. Then for every $\epsilon > 0$ there exists $T_0 > 0$ such that for every t with $e^t > T_0$ and every $v \in \mathbb{R}_+^d$ with $\|v\| > T_0$,

$$\left| C_1 e^{(d-s-2)t} \int_{K_g} f(\hat{a}_t k v) dv_g(k) - J_{f,g}(M_0^g(v), \|v\| e^{-t}) \right| < \epsilon.$$

Proof. By Lemma 2.2 of [Sar11], for all $g \in \mathcal{C}_{SL}(r_1, r_2)$ there exists a basis of \mathbb{R}^d , denoted by b_1, \dots, b_d such that

$$Q_0^g(v) = Q_{1,\dots,s}(v) + 2v_{s+1}v_d + \sum_{i=s+2}^{s+r_1} v_i^2 - \sum_{i=s+r_1+1}^{d-1} v_i^2, \quad M_0^g(v) = (v_1, \dots, v_s),$$

and

$$\hat{a}_t(v) = (v_1, \dots, v_s, e^{-t}v_{s+1}, v_{s+2}, \dots, v_{d-1}, e^t v_d),$$

where $v_i = \langle v, b_i \rangle$ for $1 \leq i \leq d$ and $Q_{1,\dots,s}(v)$ is a non degenerate quadratic form in variables v_1, \dots, v_s . Let E denote the support of f . Let $c_1 = \inf_{v \in E} \langle v, b_{s+1} \rangle$, $c_2 = \sup_{v \in E} \langle v, b_{s+1} \rangle$. From the definition of \hat{a}_t it follows that $f(\hat{a}_t w) = 0$ unless

$$(5.2) \quad |\langle w, b_{s+1} \rangle \langle w, b_d \rangle| \leq \beta,$$

$$(5.3) \quad c_1 \leq \langle w, b_{s+1} \rangle e^{-t} \leq c_2,$$

$$(5.4) \quad \pi'(w) \in \pi'(E),$$

where β depends only on E and π' denotes the projection onto the span of $b_1, \dots, b_s, b_{s+2}, \dots, b_{d-1}$. For w satisfying (5.2) and (5.3) we have $\langle w, b_d \rangle = O(e^{-t})$. This, together with (5.4) and (5.3) imply that if $f(\hat{a}_t w) \neq 0$ and t is large, then

$$(5.5) \quad \|w\| = \langle w, b_{s+1} \rangle + O(e^{-t}).$$

Note that by (5.5),

$$(5.6) \quad \langle \hat{a}_t w, b_{s+1} \rangle = \langle w, b_{s+1} \rangle e^{-t} = e^{-t} \|w\| + O(e^{-2t}),$$

and

$$(5.7) \quad \langle \hat{a}_t w, b_i \rangle = \langle w, b_i \rangle, \quad \text{for } 1 \leq i \leq s, \text{ or } s+2 \leq i \leq d-1.$$

Finally,

$$(5.8) \quad \begin{aligned} \langle \hat{a}_t w, b_d \rangle &= (Q_0^g(w) - Q_0^g(\langle w, b_1 \rangle, \dots, \langle w, b_s \rangle, 0, \langle w, b_{s+1} \rangle, \dots, \langle w, b_{d-1} \rangle, 0)) / 2 \langle \hat{a}_t w, b_{s+1} \rangle \\ &= (Q_0^g(w) - Q_0^g(\langle w, b_1 \rangle, \dots, \langle w, b_s \rangle, 0, \langle w, b_{s+1} \rangle, \dots, \langle w, b_{d-1} \rangle, 0)) / 2e^{-t} \|w\| + O(e^{-t}). \end{aligned}$$

Hence, using (5.6), (5.7) and (5.8) together with the uniform continuity of f , applied with $w = kv$ for $v \in \mathbb{R}_+^d$ and $k \in K_g$, we see that for all $\delta > 0$ there exists a $t_0 > 0$ so that if $t > t_0$ then

$$(5.9) \quad |f(\hat{a}_t kv) - f(v_1, \dots, v_s, \|v\| e^{-t}, \langle kv, b_{s+1} \rangle, \dots, \langle kv, b_{d-1} \rangle, v_d)| < \delta,$$

where v_d is determined by

$$Q_0^g(v_1, \dots, v_s, \|v\| e^{-t}, \langle kv, b_{s+1} \rangle, \dots, \langle kv, b_{d-1} \rangle, v_d) = Q_0^g(v) = a.$$

Change basis by letting $f_{s+1} = (b_{s+1} + b_d)/\sqrt{2}$, $f_d = (b_{s+1} - b_d)/\sqrt{2}$ and $f_i = b_i$ for $1 \leq i \leq s$, or $s+2 \leq i \leq d-1$. In this basis $K_g \cong SO(r_1) \times SO(r_2)$ consists of orthogonal matrices preserving the subspaces $L_1 = \langle f_1, \dots, f_s \rangle$, $L_2 = \langle f_{s+1}, \dots, f_{s+r_1} \rangle$ and $L_3 = \langle f_{s+r_1+1}, \dots, f_d \rangle$. For $i = 1, 2$ or 3 , let π_i denote the orthogonal projection onto L_i . Write $\rho_i = \|\pi_i(v)\|$; then the orbit $K_g v$ is product of a point and two spheres $\{v_1, \dots, v_s\} \times \rho_2 S^{r_1-1} \times \rho_3 S^{r_2-1}$, where S^{r_1-1} denotes the unit sphere in L_2 and S^{r_2-1} the unit sphere in L_3 .

Suppose $w \in K_g v$ is such that $f(\hat{a}_t w) \neq 0$. Then from (5.2) and (5.3) it follows that $\langle w, b_d \rangle = O(e^{-t})$. Now, set $w_i = \langle w, f_i \rangle$, then $w_{s+1} = 2^{-1/2} \langle w, b_{s+1} \rangle + O(e^{-t})$, $w_d = 2^{-1/2} \langle w, b_{s+1} \rangle + O(e^{-t})$ and for $1 \leq i \leq s$, or $s+2 \leq i \leq d-1$, $w_i = O(1)$. Hence for $i = 2$ or 3 ,

$$(5.10) \quad \rho_i = \|\pi_i(w)\| = 2^{-1/2} \langle w, b_{s+1} \rangle + O(e^{-t}) = 2^{-1/2} \|w\| + O(e^{-t}),$$

where the last estimate follows from (5.5).

By integrating (5.9) with respect to K_g we see that for all $\epsilon > 0$ there exists a $t_0 > 0$ so that if $t > t_0$ then

$$(5.11) \quad \left| \int_{K_g} f(\hat{a}_t k v) d\nu_g(k) - \int_{K_g} f(v_1, \dots, v_s, \|v\| e^{-t}, \langle k v, b_{s+1} \rangle, \dots, \langle k v, b_{d-1} \rangle, v_d) d\nu_g(k) \right| < \epsilon.$$

Equation (5.4) implies that if $f(\hat{a}_t k v) \neq 0$, then $k v$ is within a bounded distance from $\rho_2 f_{s+1} + \rho_3 f_d$. As $\|v\|$ increases so do the ρ_i and the normalised Haar measure on $\rho_2 S^{r_1-1}$ near $\rho_2 f_{s+1}$ tends to $(1/\text{Vol}(\rho_2 S^{r_1-1})) dv_{s+2} \dots dv_{s+r_1}$ and similarly the Haar measure on $\rho_3 S^{r_2-1}$ near $\rho_3 f_d$ tends to $(1/\text{Vol}(\rho_3 S^{r_2-1})) dv_{s+r_1+1} \dots dv_{d-1}$. This means that as $\|v\|$ tends to infinity the second integral in (5.11) tends to

$$(5.12) \quad \frac{\rho_2^{1-r_1} \rho_3^{1-r_2}}{C_{r_1, r_2}} \int_{\mathbb{R}^{d-s-2}} f(v_1, \dots, v_s, \|v\| e^{-t}, v_{s+1}, \dots, v_d) dv_{s+2} \dots dv_{d-1} \\ = \frac{(\|v\| e^{-t})^{d-s-2}}{\rho_2^{r_1-1} \rho_3^{r_2-1} C_{r_1, r_2}} J_{f,g}(M_0^g(v), \|v\| e^{-t}).$$

Because (5.10) implies that $\rho_2^{r_1-1} \rho_3^{r_2-1} = 2^{(s+2-d)/2} \|v\|^{d-s-2} + O(e^{-t})$ we can use (5.11) and (5.12) to get that for all $\epsilon > 0$ there exists a $t_0 > 0$ so that if $t > t_0$ and $\|v\| > t_0$ then

$$\left| \int_{K_g} f(\hat{a}_t k v) d\nu_g(k) - \frac{e^{t(s+2-d)}}{C_1} J_{f,g}(M_0^g(v), \|v\| e^{-t}) \right| < \epsilon.$$

By dividing through by the factor $\frac{e^{t(s+2-d)}}{C_1}$ we obtain the desired conclusion. \square

For f_1 and f_2 continuous functions of compact support on $\mathbb{R}_+^d = \{v \in \mathbb{R}^d : \langle v, e_{s+1} \rangle > 0\}$, define $J_{f_1, g} + J_{f_2, g} = J_{f_1+f_2, g}$ and $J_{f_1, g} J_{f_2, g} = J_{f_1 f_2, g}$. These operations make the collection of functions of the form $J_{f, g}$ into an algebra of real valued functions on the set $\mathbb{R}^s \times \{v \in \mathbb{R} : v > 0\}$. Denote by this algebra by \mathcal{A} . The following Lemma will be used in the proofs of Theorem 2.3 and Theorem 2.1.

Lemma 5.2. \mathcal{A} is dense in $C_c(\mathbb{R}^s \times \{v \in \mathbb{R} : v > 0\})$.

Proof. Let B be a compact subset of $\mathbb{R}^s \times \{v \in \mathbb{R} : v > 0\}$. Let \mathcal{A}_B denote the subalgebra of \mathcal{A} of functions with support B . It is straightforward to check that the algebra \mathcal{A}_B separates points in B and does not vanish at any point in B . Therefore, by the Stone-Weierstrass Theorem (cf. [Rud76], Theorem 7.32) \mathcal{A}_B is dense in the space of continuous functions on B . Since B is arbitrary this implies the claim. \square

Proof of Theorem 2.3. Let $\epsilon > 0$ be arbitrary and $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$. By Lemma 5.2 there exists a continuous non-negative function f on \mathbb{R}_+^d of compact support so that $J_{f, g} \geq 1 + \epsilon$ on $R \times [1, 2]$. Then

if $v \in \mathbb{R}^d$ satisfies $e^t \leq \|v\| \leq 2e^t$, $M_0^g(v) \in R$ and $Q_0^g(v) = a$ then $J_{f,g}(M_0^g(v), \|v\| e^{-t}) \geq 1 + \epsilon$. Then by Lemma 5.1, for sufficiently large t ,

$$C_1 e^{(d-s-2)t} \int_{K_g} f(\hat{a}_t k v) d\nu_g(k) \geq 1$$

if $e^t \leq \|v\| \leq 2e^t$, $M_0^g(v) \in R$, and $Q_0^g(v) = a$. Then summing over $v \in X_g(\mathbb{Z})$, we get

$$\begin{aligned} |X_g(\mathbb{Z}) \cap V_M([a, b]) \cap A(e^t, 2e^t)| &\leq \sum_{v \in X_g(\mathbb{Z})} C_1 e^{(d-s-2)t} \int_{K_g} f(\hat{a}_t k v) d\nu_g(k) \\ (5.13) \qquad \qquad \qquad &= C_1 e^{(d-s-2)t} \int_{K_g} F_{f,g}(\hat{a}_t k) d\nu_g(k). \end{aligned}$$

Note that

$$(5.14) \qquad \qquad \qquad \int_{K_g} F_{f,g}(\hat{a}_t k) d\nu_g(k) = \int_{K_I} F_{f,g}(g^{-1} a_t k g) d\nu_I(k).$$

By (2.3) we have the bound $F_{f,g}(x) \leq c(f) \alpha(x)$ for all $x \in G_g/\Gamma_g$ where $c(f)$ is a constant depending only on f . Since $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$, part I of Theorem 2.5 implies that if $r_1 \geq 3$ and $r_2 \geq 1$ then

$$(5.15) \qquad \qquad \qquad \int_{K_I} F_{f,g}(g^{-1} a_t k g) d\nu_I(k) < c(f \circ g^{-1}) \int_{K_I} \alpha(a_t k g) d\nu_I(k) < \infty.$$

In the case when $r_1 = 2$, $r_2 = 1$ and $d = 5$, the condition that $\ker(M) \cap \mathbb{Z}^d = 0$ is equivalent to the condition that $\langle e_3, e_4, e_5 \rangle \cap g\mathbb{Z}^d = 0$. Therefore, if $r_1 = 2$ and $r_2 = 1$ or 2, parts II and III of Theorem 2.5 imply that for all $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ there exists a constant C so that

$$(5.16) \qquad \qquad \qquad \int_{K_I} F_{f,g}(g^{-1} a_t k g) d\nu_I(k) < c(f \circ g^{-1}) \int_{K_I} \alpha(a_t k g) d\nu_I(k) < Ct.$$

Hence, (5.13), (5.14) and (5.15) imply that as long as $r_1 \geq 3$ and $r_2 \geq 1$ there exists a constant C_2 such that

$$|X_g(\mathbb{Z}) \cap V_M(R) \cap A(e^t, 2e^t)| \leq C_2 e^{(d-s-2)t}.$$

Similarly, (5.13), (5.14) and (5.16) imply that if $r_1 = 2$ and $r_2 = 1$ or 2, and in the case when $r_1 = 2$, $r_2 = 1$ and $d = 5$ we have $\ker(M) \cap \mathbb{Z}^d = 0$, then

$$|X_g(\mathbb{Z}) \cap V_M(R) \cap A(e^t, 2e^t)| \leq C_2 t e^{(d-s-2)t}.$$

Since we can write $T = e^t$ and

$$A(0, T) = \lim_{n \rightarrow \infty} \left(A(0, T/2^n) \bigcup_{i=1}^n A(T/2^i, T/2^{i-1}) \right),$$

the Theorem follows by summing a geometric series. \square

Theorem 2.3 has the following Corollary which is comparable with Proposition 3.7 from [EMM98] and will be used in the proof of Theorem 2.1.

Corollary 5.3. *Let f be a continuous function of compact support on \mathbb{R}_+^d . Then for every $\epsilon > 0$ and $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ there exists $t_0 > 0$ so that for $t > t_0$,*

$$(5.17) \qquad \left| e^{-(d-s-2)t} \sum_{v \in X_g(\mathbb{Z})} J_{f,g}(M_0^g(v), \|v\| e^{-t}) - C_1 \int_{K_g} F_{f,g}(\hat{a}_t k) d\nu_g(k) \right| < \epsilon.$$

Proof. Since $J_{f,g}$ has compact support, the number of non zero terms in the sum on the left hand side of (5.17) is bounded by $ce^{(d-s-2)t}$ because of Theorem 2.3. Hence summing the result of Lemma 5.1 over $v \in X_g(\mathbb{Z})$ proves (5.17). \square

5.2. Volume estimates. For a compactly supported function h on $\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\}$ we define

$$\Theta(h, T) = \int_{X_g(\mathbb{R})} h(M_0^g(v), v/T) dm_g(v).$$

For $\mathcal{X} \subseteq \mathbb{R}^d$ we will use the notation $\text{Vol}_{X_g}(\mathcal{X}) = \int_{X_g(\mathbb{R})} \mathbb{1}_{\mathcal{X} \cap X_g(\mathbb{R})} dm_g$ to mean the volume of \mathcal{X} with respect to the volume measure on $X_g(\mathbb{R})$.

The following Lemma and its Corollary are analogous to Lemma 3.8 from [EMM98] and the proofs share some similarities, although it is here that the fact we are integrating over $X_g(\mathbb{R})$ rather than the whole of \mathbb{R}^d becomes an important distinction. In Lemma 5.4 we compute $\lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(h, T)$, here it is crucial that h is not defined on $\mathbb{R}^s \times \{0\}$; if it was, using the fact that h can be bounded by an integrable function, one could directly pass the limit inside the integral and the limit would be 0. The basic strategy of Lemma 5.4 is that we evaluate the integral $\int_{X_g(\mathbb{R})} dm_g$ by switching to polar coordinates. This has the effect that the integral changes into an integral over two spheres, then we approximate the spheres by an orbit of K_g and an integral over the coordinates fixed by K_g .

Lemma 5.4. *Suppose that h is a continuous function of compact support in $\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\}$. Then*

$$\lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(h, T) = C_1 \int_{K_g} \int_0^\infty \int_{\mathbb{R}^s} h(z, rke_0) r^{d-s-2} dz \frac{dr}{2r} d\nu_g(k),$$

where e_0 is a unit vector in \mathbb{R}^d and C_1 is the constant defined by (5.1).

Proof. By Lemma 2.2 of [Sar11], for all $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ there exists a basis of \mathbb{R}^d , denoted by f_1, \dots, f_d such that

$$Q_0^g(v) = \sum_{i=1}^{s_1} v_i^2 - \sum_{i=s_1+1}^s v_i^2 + \sum_{i=s+1}^{s+r_1} v_i^2 - \sum_{i=s+r_1+1}^d v_i^2 \quad \text{and} \quad M_0^g(v) = J(v_1, \dots, v_s),$$

where $v_i = \langle v, f_i \rangle$ for $1 \leq i \leq d$, $J \in GL_s(\mathbb{R})$, s_1 is a non-negative integer such that $r_1 + s_1 = p$ and s_2 is a non-negative integer such that $r_2 + s_2 = d - p$. Let $L_1 = \langle v_1, \dots, v_{s_1}, v_{s_1+1}, \dots, v_{s+r_1} \rangle$, $L_2 = \langle v_{s_1+1}, \dots, v_s, v_{s+r_1+1}, \dots, v_d \rangle$, S^{p-1} be the unit sphere inside L_1 and S^{d-p-1} be the unit sphere inside L_2 . Let $\alpha \in S^{p-1}$ and $\beta \in S^{d-p-1}$. Using polar coordinates, we can parametrise $v \in X_g(\mathbb{R})$ so that

$$(5.18) \quad v_i = \begin{cases} \sqrt{a}\alpha_i \cosh t & \text{for } 1 \leq i \leq s_1 \\ \sqrt{a}\beta_{i-s_1} \sinh t & \text{for } s_1 + 1 \leq i \leq s \\ \sqrt{a}\alpha_{i-s+s_1} \cosh t & \text{for } s + 1 \leq i \leq s + r_1 \\ \sqrt{a}\beta_{i-s_1-r_1} \sinh t & \text{for } s + r_1 + 1 \leq i \leq d. \end{cases}$$

In these coordinates we may write

$$dm_g(v) = \frac{a^{(d-2)/2}}{2} \cosh^{p-1} t \sinh^{q-1} t dt d\xi(\alpha, \beta) = P(e^t) dt d\xi(\alpha, \beta),$$

where $P(x) = \frac{a^{(d-2)/2}}{2^{d-1}} x^{d-2} + O(x^{d-3})$ and ξ is the Haar measure on $S^{p-1} \times S^{q-1}$. Making the change of variables, $r = \frac{\sqrt{a}e^t}{2T}$, gives

$$(5.19) \quad \sqrt{a} \cosh t = Tr + a/4Tr \quad \text{and} \quad \sqrt{a} \sinh t = Tr - a/4Tr.$$

Let $L'_1 = \langle v_{s+1}, \dots, v_{s+r_1} \rangle$, $L'_2 = \langle v_{s+r_1+1}, \dots, v_d \rangle$, S^{r_1-1} be the unit sphere inside L'_1 , S^{r_2-1} be the unit sphere inside L'_2 , $\alpha' \in S^{r_1-1}$ and $\beta' \in S^{r_2-1}$. We may write

$$d\xi(\alpha, \beta) = \delta(\alpha, \beta) d\alpha_1 \dots d\alpha_{s_1} d\beta_1 \dots d\beta_{s_2} d\xi'(\alpha', \beta')$$

where $\delta(\alpha, \beta)$ is the appropriate density function and $d\xi'$ is the Haar measure on $S^{r_1-1} \times S^{r_2-1}$. This gives

$$(5.20) \quad dm_g(v) = P\left(\frac{2Tr}{\sqrt{a}}\right) \delta(\alpha, \beta) \frac{dr}{r} d\alpha_1 \dots d\alpha_{s_1} d\beta_1 \dots d\beta_{s_2} d\xi'(\alpha', \beta').$$

Let $z \in \mathbb{R}^s$. Make the further change of variables

$$(5.21) \quad (\alpha_1, \dots, \alpha_{s_1}, \beta_1, \dots, \beta_{s-s_1}) = \frac{1}{Tr} J^{-1} z,$$

this means that

$$(5.22) \quad d\alpha_1 \dots d\alpha_{s_1} d\beta_1 \dots d\beta_{s-s_1} = \frac{1}{\det(J) (Tr)^s} dz.$$

Moreover, using (5.18), (5.19) and (5.21) gives

$$(5.23) \quad M_0^g(v) = z + O(1/T) \quad \text{and} \quad v/T = r(\alpha' + \beta') + O(1/T).$$

Since h is continuous and compactly supported it may be bounded by an integrable function and hence

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(h, T) &= \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \int_{X_g(\mathbb{R})} h(M_0^g(v), v/T) dm_g(v) \\ &= \int_{X_g(\mathbb{R})} \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} h(M_0^g(v), v/T) dm_g(v) \\ &= \int_{S^{r_1-1} \times S^{r_2-1}} \int_0^\infty \int_{\mathbb{R}^s} h(z, r(\alpha' + \beta')) r^{d-s-2} \delta(\alpha', \beta') dz \frac{dr}{2r} d\xi'(\alpha', \beta'), \end{aligned}$$

where in the last step follows from (5.20), the definition of $P(x)$, (5.22) and (5.23). Note that from the definition of δ it is clear that $\delta(\alpha', \beta') = 1$. Finally, let $e_0 = \frac{1}{\sqrt{2}}(f_1 + f_{p+1})$ and $\frac{1}{\sqrt{2}}(\alpha' + \beta') = ke_0$ and $r' = \sqrt{2}r$ to get that

$$\lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(h, T) = C_1 \int_{K_g} \int_0^\infty \int_{\mathbb{R}^s} h(z, r'ke_0) r'^{d-s-2} dz \frac{dr'}{2r'} d\nu_g(k).$$

□

Corollary 5.5. *For all $g \in \mathcal{C}_{SL}(r_1, r_2)$ there exists a constant $C_3 > 0$ such that for all compact regions $R \subset \mathbb{R}^s$ with piecewise smooth boundary*

$$\lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \text{Vol}_{X_g} \left(V_{M_0^g}(R) \cap A(T/2, T) \right) = C_3 \text{Vol}(R).$$

Proof. Let $\mathbb{1}$ denote the characteristic function of $R \times A(1/2, 1)$, then it is clear that

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \text{Vol}_{X_g} \left(V_{M_0^g}(R) \cap A(T/2, T) \right) &= \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \int_{X_g(\mathbb{R})} \mathbb{1}(M_0(gv), v/T) dm_g(v) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(\mathbb{1}, T). \end{aligned}$$

Since R is connected and has smooth boundary there exists regions $R_\delta^- \subseteq R \times A(1/2, 1) \subseteq R_\delta^+$ such that $\lim_{\delta \rightarrow 0} R_\delta^+ = \lim_{\delta \rightarrow 0} R_\delta^- = R$ and for all $\delta > 0$ we can choose continuous compactly supported functions h_δ^- and h_δ^+ on $\mathbb{R}^s \times \mathbb{R}^d \setminus 0$, such that $0 \leq h_\delta^- \leq \mathbb{1} \leq h_\delta^+ \leq 1$, $h_\delta^-(v) = \mathbb{1}(v)$ if $v \in R_\delta^-$ and $h_\delta^+(v) = 0$ if $v \notin R_\delta^+$. By Lemma 5.4

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(h_\delta^-, T) &\leq \liminf_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \int_{X_g(\mathbb{R})} \mathbb{1}(M_0(gv), v/T) dm_g(v) \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \int_{X_g(\mathbb{R})} \mathbb{1}(M_0(gv), v/T) dm_g(v) \leq \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(h_\delta^+, T). \end{aligned}$$

It is clear that

$$\lim_{\delta \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(h_\delta^-, T) = \lim_{\delta \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(h_\delta^+, T) = \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(\mathbb{1}, T),$$

hence we can apply Lemma 5.4 to get that

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(1, T) &= C_1 \int_{K_g} \int_0^\infty \int_{\mathbb{R}^s} \mathbb{1}(z, rk^{-1}e_0) r^{d-s-2} dz \frac{dr}{2r} d\nu_g(k) \\ &= C_1 \int_{\mathbb{R}^s} \mathbb{1}_R(z) dz \int_{K_g} \int_0^\infty \mathbb{1}_{A(1/2,1)}(rk^{-1}e_0) r^{d-s-2} \frac{dr}{2r} d\nu_g(k) = C_3 \text{Vol}(R). \end{aligned}$$

The last equality holds because

$$\int_{K_g} \int_0^\infty \mathbb{1}_{A(1/2,1)}(rk^{-1}e_0) r^{d-s-2} \frac{dr}{2r} d\nu_g(k) < \infty$$

as $\mathbb{1}_{A(1/2,1)}$ has compact support and K_g is compact. \square

5.3. Proof of Theorem 2.1. By Theorem 4.9 of [PR94] there exists $v_1, \dots, v_j \in X_g(\mathbb{Z})$ such that $X_g(\mathbb{Z}) = \bigsqcup_{i=1}^j \Gamma_g v_i$. Let $P_i(g) = \{x \in G_g : xv_i = v_i\}$ and $\Lambda_i(g) = P_i(g) \cap \Gamma_g$. By Proposition 1.13 of [Hel00] there exists Haar measures ϱ_{Λ_i} , p_{Λ_i} and γ_{Λ_i} on $G_g/\Lambda_i(g)$, $P_i(g)/\Lambda_i(g)$ and $\Gamma_g/\Lambda_i(g)$ respectively such that, for $f \in C_c(G_g/\Lambda_i(g))$, and hence for integrable functions on $G_g/\Lambda_i(g)$,

$$(5.24) \quad \int_{G_g/\Lambda_i(g)} f d\varrho_{\Lambda_i} = \int_{X_g(\mathbb{R})} \int_{P_i(g)/\Lambda_i(g)} f(xp) dp_{\Lambda_i}(p) dm_g(x),$$

and

$$(5.25) \quad \int_{G_g/\Lambda_i(g)} f d\varrho_{\Lambda_i} = \int_{G_g/\Gamma_g} \int_{\Gamma_g/\Lambda_i(g)} f(x\gamma) d\gamma_{\Lambda_i}(\gamma) d\mu_g(x).$$

Note that $\Gamma_g/\Lambda_i(g) = \Gamma_g v_i$ is discrete and its Haar measure $d\gamma_{\Lambda_i}$ is just the counting measure and so

$$(5.26) \quad \int_{\Gamma_g/\Lambda_i(g)} f(x\gamma) d\gamma_{\Lambda_i}(\gamma) = \sum_{v \in \Gamma_g v_i} f(xv).$$

Therefore the normalisations already present on m_g and μ_g induce a normalisation on n_{Λ_i} . Moreover, it follows from the Borel Harish-Chandra Theorem (cf. [PR94], Theorem 4.13) that the measure of $p_{\Lambda_i}(P_i(g)/\Lambda_i(g)) < \infty$, for each $1 \leq i \leq j$. As in [EMM98] and [DM93] where the proofs rely on Siegel's integral formula, here the proof relies on the following result.

Lemma 5.6. *For all $f \in C_c(X_g(\mathbb{R}))$ and $g \in \mathcal{C}_{SL}(r_1, r_2)$ there exists a constant*

$$C(g) = \sum_{i=1}^j p_{\Lambda_i}(P_i(g)/\Lambda_i(g)),$$

such that

$$(5.27) \quad C(g) \int_{X_g(\mathbb{R})} f dm_g = \int_{G_g/\Gamma_g} F_{f,g} d\mu_g.$$

Proof. Note that for $1 \leq i \leq j$, $G_g/P_i(g) \cong X_g(\mathbb{R})$. If $f \in C_c(X_g(\mathbb{R}))$ then f is $\Lambda_i(g)$ invariant and therefore can be considered as an integrable function on $G_g/\Lambda_i(g)$ and so

$$(5.28) \quad \int_{X_g(\mathbb{R})} \int_{P_i(g)/\Lambda_i(g)} f(xp) dp_{\Lambda_i}(p) dm_g(x) = \int_{P_i(g)/\Lambda_i(g)} dp_{\Lambda_i} \int_{X_g(\mathbb{R})} f dm_g.$$

Now it follows from the definition of $F_{f,g}$ (i.e. (2.2)), (5.24), (5.25), (5.26) and (5.28) that

$$\begin{aligned} \int_{G_g/\Gamma_g} F_{f,g} d\mu_g &= \sum_{i=1}^j \int_{G_g/\Gamma_g} \sum_{v \in \Gamma_g v_i} f(xv) d\mu_g(x) \\ &= \sum_{i=1}^j \int_{P_i(g)/\Lambda_i(g)} dp_{\Lambda_i} \int_{X_g(\mathbb{R})} f dm_g, \end{aligned}$$

which is the desired result. \square

The final Lemma of this section is the counterpart of Lemma 3.9 from [EMM98] and again the proof there is mimicked.

Lemma 5.7. *Let f be a continuous function of compact support on \mathbb{R}_+^d . Then for all $g \in \mathcal{C}_{SL}(r_1, r_2)$,*

$$\lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \int_{X_g(\mathbb{R})} J_{f,g}(M_0^g(v), \|v\|/T) dm_g(v) = C_1 C(g) \int_{G_g/\Gamma_g} F_{f,g} d\mu_g,$$

where C_1 is defined by (5.1) and $C(g)$ is defined in Lemma 5.6.

Proof. Let v_i be the components of v when written in the basis b_1, \dots, b_d from Lemma 5.1. Using the change of variables $(v_1, \dots, v_d) \rightarrow (z_1, \dots, z_s, r, v_{s+2}, \dots, v_d)$ where $Q_0^g(v_1, \dots, v_d) = a$ we see that

$$\int_{\mathbb{R}^d} f(v) dv = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{\mathbb{R}^s} J_{f,g}(z, r) r^{d-s-2} dz \frac{dr}{2r} da.$$

Hence it follows from how m_g is defined (i.e. (2.4)) that

$$(5.29) \quad \int_{X_g(\mathbb{R})} f(v) dm_g(v) = \int_0^{\infty} \int_{\mathbb{R}^s} J_{f,g}(z, r) r^{d-s-2} dz \frac{dr}{2r}.$$

Lemma 5.4 and (5.29) imply that

$$\lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \int_{X_g(\mathbb{R})} J_{f,g}(M_0^g(v), \|v\|/T) dm_g(v) = C_1 \int_{K_g} \left(\int_{X_g(\mathbb{R})} f(v) dm_g \right) d\nu_g(k).$$

Now the conclusion follows from Lemma 5.6. \square

The purpose of Lemma 5.7 is to relate the integral over G_g/Γ_g to an integral over $X_g(\mathbb{R})$ in order that the integral over $X_g(\mathbb{R})$ can be approximated by an integral over K_g via Theorem 2.7. Then the integral over K_g can be approximated by the appropriate counting function via Corollary 5.3. We now proceed to put this into action in the proof of our main Theorem which is just a modification of the proof in [EMM98].

Proof of Theorem 2.1. By Lemma 5.4 the functional Ψ on $C_c(\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\})$ given by

$$\Psi(h) = \lim_{T \rightarrow \infty} \frac{1}{T^{d-s-2}} \Theta(h, T)$$

is continuous. For all connected regions $R \subset \mathbb{R}^s$ with smooth boundary, if $\mathbb{1}$ denotes the characteristic function of $R \times A(1/2, 1)$, then for every $\epsilon > 0$ there exist continuous functions h_+ and h_- on $\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\}$ such that for all $(r, v) \in \mathbb{R}^s \times \mathbb{R}^d \setminus \{0\}$,

$$(5.30) \quad h_-(r, v) \leq \mathbb{1}(r, v) \leq h_+(r, v)$$

and

$$(5.31) \quad |\Psi(h_+) - \Psi(h_-)| < \epsilon.$$

Let \mathcal{J} denote the space of linear combinations of functions on $\mathbb{R}^s \times \mathbb{R}^d$ of the form $J_{f,g}(r, \|v\|)$, where f is continuous function of compact support on \mathbb{R}_+^d . Let \mathcal{H} denote the collection of functions in $C_c(\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\})$ such that if $h \in \mathcal{H}$ then h takes an argument of the form $(r, \|v\|)$. By Lemma 5.2 \mathcal{J} is dense in \mathcal{H} and since h_+ and h_- belong to \mathcal{H} we may suppose that h_+ and h_- may be written as a finite linear combination of functions from \mathcal{J} . The function $F_{f,g}$ defined by (2.2) obeys the bound (2.5) with $\delta = 1$, by (2.3). Moreover, Lemma 3.10 of [EMM98] implies that $F_{f,g} \in L_1(G_g/\Gamma_g)$. Therefore, if $h' \in \{h_+, h_-\}$, then for all $g \in \mathcal{C}_{SL}(r_1, r_2)$ we can apply Theorem 2.7 with the function $F_{f,g}$, followed by Corollary 5.3 and Lemma 5.7 to get that there exists $t_0 > 0$, so that for all $\epsilon > 0$ and $t > t_0$,

$$(5.32) \quad \left| \frac{C(g)}{e^{(d-s-2)t}} \sum_{v \in X_g(\mathbb{Z})} h'(M_0^g(v), ve^{-t}) - \Psi(h') \right| < \epsilon.$$

From the definition of $\Psi(h)$ we see that for all $h \in C_c(\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\})$ and $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ there exists $t_0 > 0$, so that for all $\epsilon > 0$ and $t > t_0$,

$$(5.33) \quad \left| \frac{1}{e^{(d-s-2)t}} \int_{X_g(\mathbb{R})} h(M_0^g(v), ve^{-t}) dm_g(v) - \Psi(h) \right| < \epsilon.$$

Clearly (5.30) implies

$$(5.34) \quad \begin{aligned} \frac{C(g)}{e^{(d-s-2)t}} \sum_{v \in X_g(\mathbb{Z})} h_-(M_0^g(v), ve^{-t}) - \Psi(h_+) &\leq \frac{C(g)}{e^{(d-s-2)t}} \sum_{v \in X_g(\mathbb{Z})} \mathbb{1}(M_0^g(v), ve^{-t}) - \Psi(h_+) \\ &\leq \frac{C(g)}{e^{(d-s-2)t}} \sum_{v \in X_g(\mathbb{Z})} h_+(M_0^g(v), ve^{-t}) - \Psi(h_+). \end{aligned}$$

Apply (5.31) to the left hand side of (5.34) and then apply (5.32) with suitable choices of ϵ 's to get that for all $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$,

$$(5.35) \quad \left| \frac{C(g)}{e^{(d-s-2)t}} \sum_{v \in X_g(\mathbb{Z})} \mathbb{1}(M_0^g(v), ve^{-t}) - \Psi(h_+) \right| \leq \frac{\theta}{2}.$$

Similarly using (5.30), (5.31) and (5.33) we see that for all $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$,

$$(5.36) \quad \left| \frac{1}{e^{(d-s-2)t}} \int_{X_g(\mathbb{R})} \mathbb{1}(M_0^g(v), ve^{-t}) dm_g(v) - \Psi(h_+) \right| \leq \frac{\theta}{2}.$$

Hence using (5.35) and (5.36) we see that for all $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$

$$(5.37) \quad \left| C(g) \sum_{v \in X_g(\mathbb{Z})} \mathbb{1}(M_0^g(v), ve^{-t}) - \int_{X_g} \mathbb{1}(M_0^g(v), ve^{-t}) dm_g(v) \right| \leq \theta.$$

This means that for all $g \in \mathcal{C}_{\text{SL}}(r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$,

$$(5.38) \quad \begin{aligned} (1 - \theta) \int_{X_g(\mathbb{R})} \mathbb{1}(M_0^g(v), ve^{-t}) dm_g(v) \\ \leq C(g) \sum_{v \in X_g(\mathbb{Z})} \mathbb{1}(M_0^g(v), ve^{-t}) \leq (1 + \theta) \int_{X_g(\mathbb{R})} \mathbb{1}(M_0^g(v), ve^{-t}) dm_g(v). \end{aligned}$$

Hence for all $(Q, M) \in \mathcal{C}_{\text{Pairs}}(r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$,

$$\begin{aligned} (1 - \theta) \text{Vol}_{X_Q}(V_M(R) \cap A(T/2, T)) &\leq C(g) |X_Q(\mathbb{Z}) \cap V_M(R) \cap A(T/2, T)| \\ &\leq (1 + \theta) \text{Vol}_{X_Q}(V_M(R) \cap A(T/2, T)). \end{aligned}$$

The conclusion of the Theorem follows by applying Corollary 5.5 and summing a geometric series. \square

6. COUNTEREXAMPLES

In small dimensions there are slightly more integer points than expected on the quadratic surfaces defined by forms with signature $(1, 2)$ and $(2, 2)$. This fact was exploited in [EMM98] to show that the expected asymptotic formula for the situation they consider is not valid for these special cases. In a similar manner it is possible to construct examples that show that Theorem 1.1 is not valid in the cases that the signature of H_g is $(1, 2)$ or $(2, 2)$. In this section, for the sake of brevity we restrict our

attention to the case when $s = 1$, but we note that similar arguments would hold in the case when $s > 1$. To start with make the following definitions

$$\begin{aligned} Q_1(x) &= -x_1x_2 + x_3^2 + x_4^2, \\ Q_2(x) &= x_1x_2 + x_3^2 - x_4^2, \\ Q_3(x) &= -x_1x_2 + x_3^2 + x_4^2 - \alpha x_5^2, \\ L_\alpha(x) &= x_1 - \alpha x_2. \end{aligned}$$

We can now prove.

Lemma 6.1. *Let $\epsilon > 0$, suppose $[a, b] = [1/2 - \epsilon, 1]$ or $[-1, -1/2 + \epsilon]$. Let $a > 0$, then for every $T_0 > 0$, the set of $\beta \in \mathbb{R}$ for which there exists a $T > T_0$ such that*

$$|X_{Q_1}^a(\mathbb{Z}) \cap V_{L_\beta}([a, b]) \cap A(0, T)| > T(\log T)^{1-\epsilon} \quad \text{or} \quad |X_{Q_2}^a(\mathbb{Z}) \cap V_{L_\beta}([a, b]) \cap A(0, T)| > T(\log T)^{1-\epsilon}$$

is dense. Similarly if $a = 0$, then for every $T_0 > 0$, the set of $\beta \in \mathbb{R}$ for which there exists a $T > T_0$ such that

$$|X_{Q_3}^a(\mathbb{Z}) \cap V_{L_\beta}([a, b]) \cap A(0, T)| > T^2(\log T)^{1-\epsilon}$$

is dense.

Proof. Let $S_i(\alpha, T, a) = \{x \in \mathbb{Z}^{d_i} : L_\alpha(x) = 0, Q_i(x) = a, \|x\| \leq T\}$ where $d_i = 4$ if $i = 1$ or 2 and $d_i = 5$ if $i = 3$. Lemma 3.14 of [EMM98] implies that

$$(6.1) \quad |S_i(\alpha, T, a)| \sim T \log T \quad \text{for } i = 1, 2 \text{ and } \sqrt{\alpha} \in \mathbb{Q} \text{ and } a > 0,$$

$$(6.2) \quad |S_3(\alpha, T, 0)| \sim T^2 \log T \quad \text{for } \sqrt{\alpha} \in \mathbb{Q}.$$

Note that if $i = 1, 2$ and $x \in S_i(\alpha, T, a) \setminus S_i(\alpha, T/2, a)$, then

$$(6.3) \quad \frac{T^2}{4} - (\alpha^2 + 1)x_2^2 \leq x_3^2 + x_4^2 \leq T^2 - (\alpha^2 + 1)x_2^2$$

and

$$(6.4) \quad x_3^2 + x_4^2 = \alpha x_2^2 + a.$$

Similarly if $x \in S_3(\alpha, T, 0) \setminus S_3(\alpha, T/2, 0)$,

$$(6.5) \quad \frac{T^2}{4} - (\alpha^2 + 1)x_2^2 \leq x_3^2 + x_4^2 + x_5^2 \leq T^2 - (\alpha^2 + 1)x_2^2$$

and

$$(6.6) \quad x_3^2 + x_4^2 = \alpha(x_2^2 + x_5^2).$$

Combining (6.3) and (6.4) gives

$$(6.7) \quad \frac{T^2 - 4a}{4(\alpha^2 + \alpha + 1)} \leq x_2^2 \leq \frac{T^2 - a}{\alpha^2 + \alpha + 1}.$$

Respectively, combining (6.5) and (6.6) gives

$$(6.8) \quad \frac{T^2 - (\alpha + 1)x_5^2}{4(\alpha^2 + \alpha + 1)} \leq x_2^2 \leq \frac{T^2 - (\alpha + 1)x_5^2}{\alpha^2 + \alpha + 1},$$

which upon noting that $-T \leq x_5 \leq T$ offers

$$(6.9) \quad \frac{T^2 - (\alpha + 1)T}{4(\alpha^2 + \alpha + 1)} \leq x_2^2 \leq \frac{T^2 + (\alpha + 1)T}{\alpha^2 + \alpha + 1}.$$

Take

$$(6.10) \quad \beta_\pm = \alpha \pm \sqrt{\frac{\alpha^2 + \alpha + 1}{T^2}}.$$

It is clear that $L_{\beta_{\pm}}(x) = L_{\alpha}(x) \pm \sqrt{\frac{\alpha^2 + \alpha + 1}{T^2}}x_2$ and hence if $i = 1, 2$ and $x \in S_i(\alpha, T, a) \setminus S_i(\alpha, T/2, a)$, then (6.7) implies

$$(6.11) \quad \sqrt{\frac{1}{4} - \frac{a}{T^2}} \leq L_{\beta_+}(x) \leq \sqrt{1 - \frac{a}{T^2}} \quad \text{and} \quad -\sqrt{1 - \frac{a}{T^2}} \leq L_{\beta_-}(x) \leq -\sqrt{\frac{1}{4} - \frac{a}{T^2}}.$$

Similarly if $x \in S_3(\alpha, T, 0) \setminus S_3(\alpha, T/2, 0)$, then (6.9) implies

$$(6.12) \quad \sqrt{\frac{1}{4} - \frac{(\alpha + 1)}{T}} \leq L_{\beta_+}(x) \leq \sqrt{1 - \frac{(\alpha + 1)}{T}} \quad \text{and} \quad -\sqrt{1 - \frac{(\alpha + 1)}{T}} \leq L_{\beta_-}(x) \leq -\sqrt{\frac{1}{4} - \frac{(\alpha + 1)}{T}}.$$

This means for all $\epsilon > 0$ there exists $T_+ > 0$ such that if $T > T_+$ then $S_i(\alpha, T, a) \subset X_{Q_i}^a(\mathbb{Z}) \cap V_{L_{\beta_+}}([1/2 - \epsilon, 1]) \cap A(0, T)$ respectively there also exists $T_- > 0$ such that if $T > T_-$ then $S_i(\alpha, T, a) \subset X_{Q_i}^a(\mathbb{Z}) \cap V_{L_{\beta_-}}([-1, -1/2 + \epsilon]) \cap A(0, T)$. By (6.1) and (6.2) for $i = 1, 2$ and large enough T , $|S_i(\alpha, T, a)| > T(\log T)^{1-\epsilon}$ and $|S_i(\alpha, T, a)| > CT^2(\log T)^{1-\epsilon}$. The set of β satisfying (6.10) for rational α and large T is clearly dense and this proves the Lemma. \square

Theorem 6.2. *Let $j = 1, 2$. For every $\epsilon > 0$ and every interval $[a, b]$ there exists a rational quadratic form Q and an irrational linear form L such that $\text{Stab}_{SO(Q)}(L) \cong SO(j, 2)$ such that for an infinite sequence $T_k \rightarrow \infty$,*

$$\left| X_Q^{a_j}(\mathbb{Z}) \cap V_L([a, b]) \cap A(0, T_k) \right| > T_k^j (\log T_k)^{1-\epsilon},$$

where $a_1 > 0$ and $a_2 = 0$.

Proof. Since the interval $[a, b]$ must intersect either the positive or negative reals there is no loss of generality in assuming, after passing to a subset and rescaling that $[a, b] = [1/4, 5/4]$ or $[-5/4, -1/4]$. For a given $S > 0$ and $i = 1, 2$ let \mathcal{U}_S be the set of $\gamma \in \mathbb{R}$ for which there exists $\beta \in \mathbb{R}$ and $T > S$ with

$$(6.13) \quad \left| X_{Q_i}^{a_i}(\mathbb{Z}) \cap V_{L_{\beta}}([1/2, 1]) \cap A(0, T) \right| > CT \log T,$$

and

$$(6.14) \quad |\beta - \gamma| < T^{-2}.$$

Then \mathcal{U}_S is open and dense by Lemma 6.1. By the Baire category Theorem (cf. [Rud87], Theorem 5.6) $\bigcap_{k=1}^{\infty} \mathcal{U}_{2^{k+1}}$ is dense in \mathbb{R} and is in fact of second category and hence uncountable. Let $\gamma \in \bigcap_{k=1}^{\infty} \mathcal{U}_{2^{k+1}} \setminus \mathbb{Q}$, then there exists infinite sequences β_k and T_k such that (6.13) and (6.14) hold with β replaced by β_k and T by T_k . Note that (6.14) implies that for $\|x\| < T_k$,

$$|L_{\beta_k}(x) - L_{\gamma}(x)| < \frac{1}{T_k} < \frac{1}{4},$$

so that

$$X_{Q_i}^{a_i}(\mathbb{Z}) \cap V_{L_{\beta_k}}([1/2, 1]) \cap A(0, T_k) \subseteq X_{Q_i}^{a_i}(\mathbb{Z}) \cap V_{L_{\gamma}}([1/4, 5/4]) \cap A(0, T_k)$$

and hence $\left| X_{Q_i}^{a_i}(\mathbb{Z}) \cap V_{L_{\gamma}}([1/4, 5/4]) \cap A(0, T_k) \right| > CT_k \log T_k$ by (6.13). If $i = 3$ then we can carry out the same process but we replace \mathcal{U}_S by the set \mathcal{W}_S , of $\gamma \in \mathbb{R}$ for which there exists $\beta \in \mathbb{R}$ and $T > S$ with

$$\left| X_{Q_3}^0(\mathbb{Z}) \cap V_{L_{\beta}}([1/2, 1]) \cap A(0, T) \right| > CT^2 \log T,$$

and

$$|\beta - \gamma| < T^{-2}.$$

\square

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